On the Expressive Power of Weighted Restarting Automata

Qichao Wang

Fachbereich Elektrotechnik/Informatik, Universität Kassel
34109 Kassel, Germany

9th Workshop on Non-Classical Models of Automata and Applications

August 17-18, 2017, Prague, Czech

Introduction

Motivation

- Analysis by Reduction
  - checking the correctness of sentences of natural languages

Introduction

Motivation

- Analysis by Reduction
  - checking the correctness of sentences of natural languages

Previous Works

- Restarting Automata [Jancar, Mráz, Plátek, Vogel, 1995]
  - a formal model for analysis by reduction

Introduction

Motivation

- Analysis by Reduction
  - checking the correctness of sentences of natural languages

Previous Works

- Restarting Automata [Jancar, Mráz, Plátek, Vogel, 1995]
  - a formal model for analysis by reduction
- Weighted Restarting Automata [Otto, Wang, 2014]
  - studying quantitative aspects of computations of restarting automata

Introduction

Motivation

- Analysis by Reduction
  - checking the correctness of sentences of natural languages

Previous Works

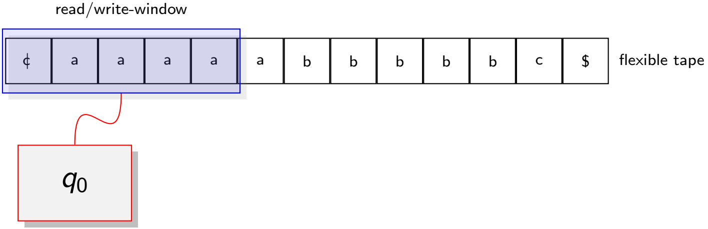
- Restarting Automata [Jancar, Mráz, Plátek, Vogel, 1995]
  - a formal model for analysis by reduction
- Weighted Restarting Automata [Otto, Wang, 2014]
  - studying quantitative aspects of computations of restarting automata

Question

- Which languages can weighted restarting automata accept?

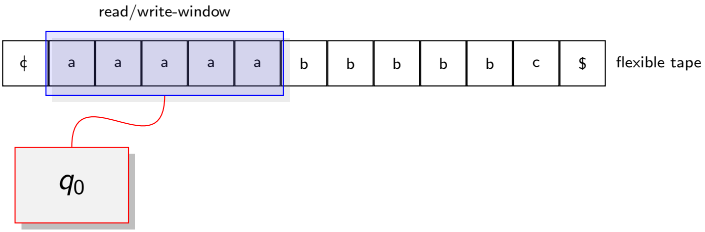
Restarting Automata (RRWW-Automata)

M_1 :

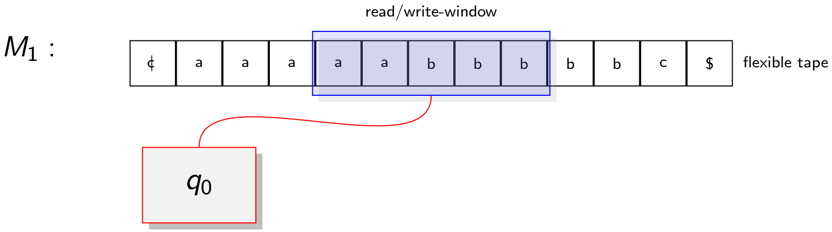


Restarting Automata (RRWW-Automata)

M_1 :

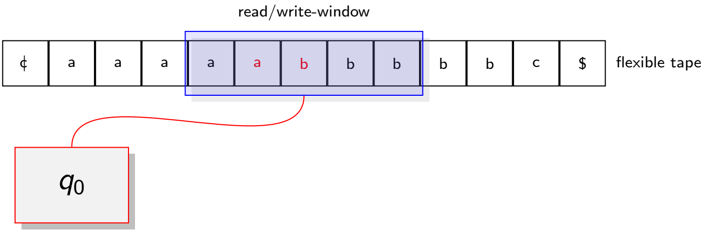


Restarting Automata (RRWW-Automata)



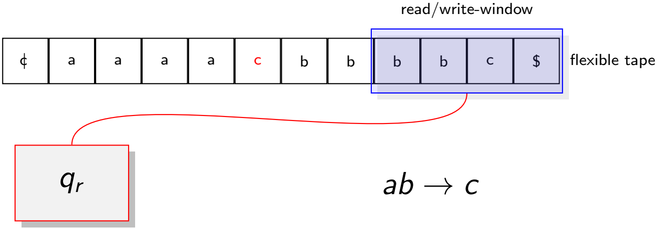
Restarting Automata (RRWW-Automata)

M_1 :



Restarting Automata (RRWW-Automata)

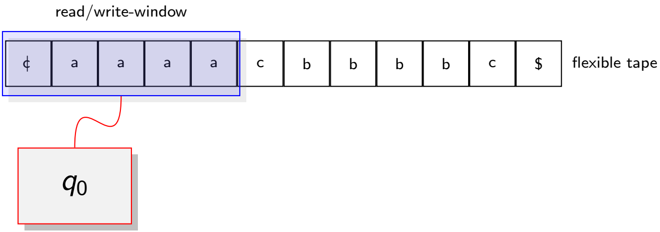
M_1 :



$ab \to c$

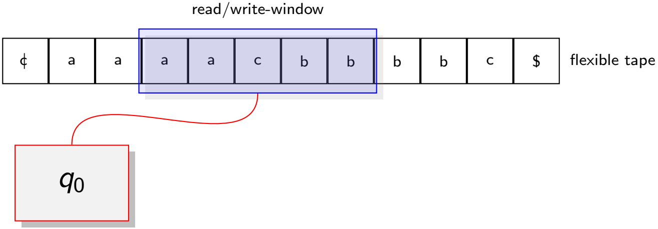
Restarting Automata (RRWW-Automata)

M_1 :

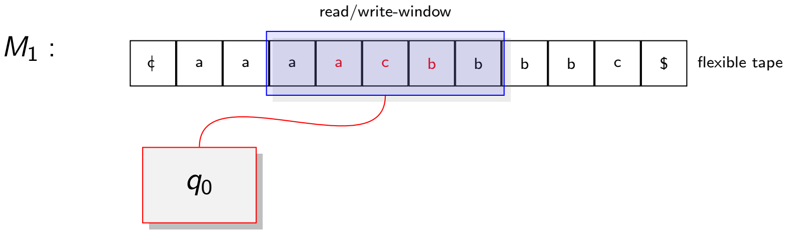


Restarting Automata (RRWW-Automata)

M_1 :

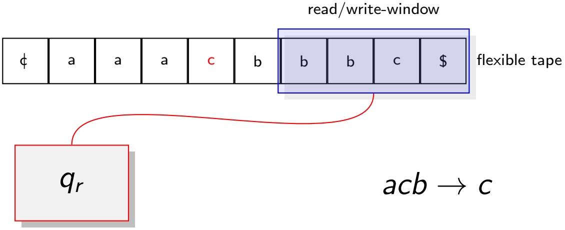


Restarting Automata (RRWW-Automata)



Restarting Automata (RRWW-Automata)

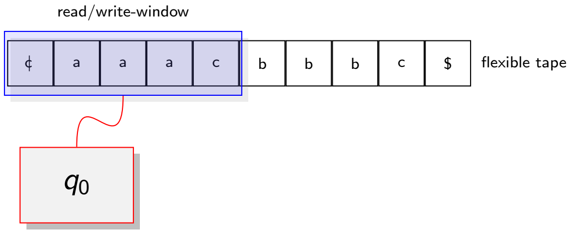
M_1 :



$$acb \to c$$

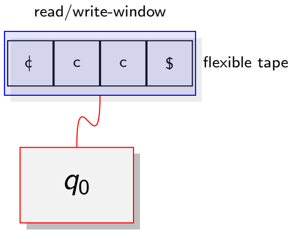
Restarting Automata (RRWW-Automata)

M_1 :



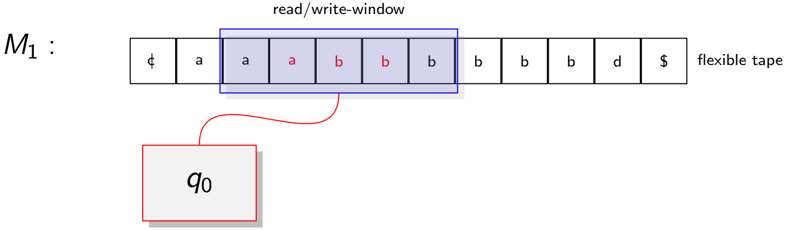
Restarting Automata (RRWW-Automata)

M_1 :



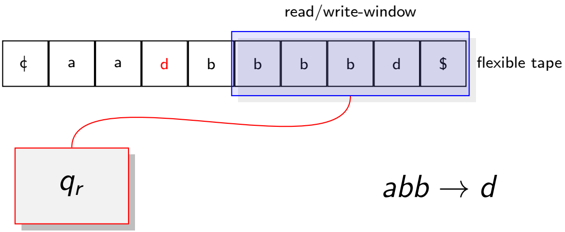
Accept

Restarting Automata (RRWW-Automata)



Restarting Automata (RRWW-Automata)

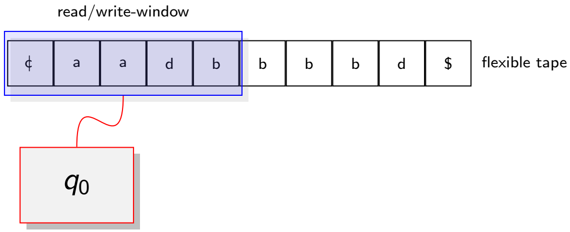
M_1 :



$abb \to d$

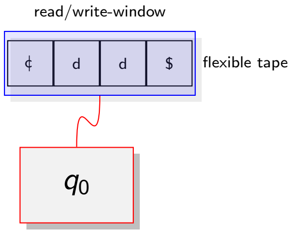
Restarting Automata (RRWW-Automata)

M_1 :



Restarting Automata (RRWW-Automata)

M_1 :



Accept

Meta-Instructions of M_1

$$
\begin{array}{llllllll}
t_1: & (\cent a^*, & ab & \to & c, & b^*c\$), & t_5: & (\cent cc\$, \quad \text{Accept}),\\
t_2: & (\cent a^*, & acb & \to & c, & b^*c\$), & t_6: & (\cent dd\$, \quad \text{Accept}),\\
t_3: & (\cent a^*, & abb & \to & d, & b^*d\$), & t_7: & (\cent c\$, \quad \text{Accept}),\\
t_4: & (\cent a^*, & adbb & \to & d, & b^*d\$), & t_8: & (\cent d\$, \quad \text{Accept}).
\end{array}
$$

Meta-Instructions of M_1

$$
\begin{array}{llllll}
t_1: & (¢a^*, & ab \to c, & b^*c\$), & t_5: & (¢cc\$, \text{ Accept}),\\
t_2: & (¢a^*, & acb \to c, & b^*c\$), & t_6: & (¢dd\$, \text{ Accept}),\\
t_3: & (¢a^*, & abb \to d, & b^*d\$), & t_7: & (¢c\$, \text{ Accept}),\\
t_4: & (¢a^*, & adbb \to d, & b^*d\$), & t_8: & (¢d\$, \text{ Accept}).
\end{array}
$$

The language Accepted by M_1

- $L(M_1) = L_1 \cup L_2$, where
  - $L_1 = \{ a^n cb^n c, a^n db^{2n} d \mid n \geq 0 \}$ and $L_2 = \{ a^n b^n c, a^n b^{2n} d \mid n \geq 0 \}$.

Definition 1

A *weighted restarting automaton* is described by a couple (M, ω), where

- M is a restarting automaton on some input alphabet Σ,
- ω is a weight function that assigns a weight from some semiring S to each transition of M.

Definition 1

A *weighted restarting automaton* is described by a couple (M, ω), where

- M is a restarting automaton on some input alphabet Σ,
- ω is a weight function that assigns a weight from some semiring S to each transition of M.

Definition 2

The *weight* of an accepting computation $C = (c_0 \vdash_M c_1 \vdash_M \cdots \vdash_M c_n)$ is defined by

$$weight(C) = \prod_{0 \leq i \leq n-1} \omega(t(c_i, c_{i+1})),$$

where $t(c_i, c_{i+1})$ is the transition from the configuration c_i to the configuration c_{i+1}.

Definition 3

The *behavior* of a weighted restarting automaton (M, ω) is the function $f_\omega^M : \Sigma^* \to S$ defined by

$$f_\omega^M(w) = \sum_{C \in C_M(w)} weight(C),$$

where $C_M(w)$ is the set of accepting computations of M on the input w.

Definition 3

The *behavior* of a weighted restarting automaton (M, ω) is the function $f_\omega^M : \Sigma^* \to S$ defined by

$$f_\omega^M(w) = \sum_{C \in C_M(w)} weight(C),$$

where $C_M(w)$ is the set of accepting computations of M on the input w.

Example 1: using the semiring $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$

$\mathcal{M}_1 = (M_1, \omega_1)$, where

$$\omega_1(t) = \begin{cases} 1, & \text{if } t \text{ is an accept transition,} \\ & \text{or a rewrite transition: } ab \to c \text{ or } abb \to d, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\omega_1}^{M_1}(w) = \begin{cases} 1, & \text{if } w \in L_1 = \{\, a^n c b^n c, a^n d b^{2n} d \mid n \geq 0 \,\}, \\ 2, & \text{if } w \in L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}, \\ 0, & \text{otherwise.} \end{cases}$$

Weighted Restarting Automata as Language Acceptors

Definition 4

Let $M = (Q, \Sigma, \Gamma, ¢, \$, q_0, k, \delta)$ be a restarting automaton, let ω be a weight function from M into a semiring S, and let $\mathcal{M} = (M, \omega)$. For a subset T of S, $L_T(\mathcal{M}) = \{ w \in L(M) \mid f_\omega^M(w) \in T \}$ is the *language accepted by M relative to T*, that is, a word $w \in \Sigma^*$ belongs to the language $L_T(\mathcal{M})$ iff $w \in L(M)$ and $f_\omega^M(w) \in T$.

Weighted Restarting Automata as Language Acceptors

Definition 4

Let $M = (Q, \Sigma, \Gamma, ¢, \$, q_0, k, \delta)$ be a restarting automaton, let ω be a weight function from M into a semiring S, and let $\mathcal{M} = (M, \omega)$. For a subset T of S, $L_T(\mathcal{M}) = \{ w \in L(M) \mid f_\omega^M(w) \in T \}$ is the *language accepted by $\mathcal{M}$ relative to T*, that is, a word $w \in \Sigma^*$ belongs to the language $L_T(\mathcal{M})$ iff $w \in L(M)$ and $f_\omega^M(w) \in T$.

Definition 5

Let X be a type of restarting automaton, let S be a semiring, and let $\mathbb{H}$ be a family of subsets of S. Then

$$\mathcal{L}(\mathsf{X}, S, \mathbb{H}) = \{ L_T(\mathcal{M}) \mid \begin{array}{l} \mathcal{M} \text{ is a weighted restarting automaton} \\ \text{of type X, and } T \in \mathbb{H} \end{array} \}$$

is the class of languages that are accepted by weighted restarting automata of type X relative to $\mathbb{H}$.

Example 1 (Cont.)

$\mathcal{M}_1 = (M_1, \omega_1)$, where

$$\omega_1(t) = \begin{cases} 1, & \text{if } t \text{ is an accept transition,} \\ & \text{or a rewrite transition: } ab \to c \text{ or } abb \to d, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\omega_1}^{M_1}(w) = \begin{cases} 1, & \text{if } w \in L_1 = \{\, a^n c b^n c, a^n d b^{2n} d \mid n \geq 0 \,\}, \\ 2, & \text{if } w \in L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}, \\ 0, & \text{otherwise.} \end{cases}$$

- Let $\mathcal{M}_1 = (M_1, \omega_1)$, and let $T_1 = \{\, 2 \,\}$.

.

Example 1 (Cont.)

$\mathcal{M}_1 = (M_1, \omega_1)$, where

$$\omega_1(t) = \begin{cases} 1, & \text{if } t \text{ is an accept transition,} \\ & \text{or a rewrite transition: } ab \to c \text{ or } abb \to d, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\omega_1}^{M_1}(w) = \begin{cases} 1, & \text{if } w \in L_1 = \{\, a^n c b^n c, a^n d b^{2n} d \mid n \geq 0 \,\}, \\ 2, & \text{if } w \in L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}, \\ 0, & \text{otherwise.} \end{cases}$$

- Let $\mathcal{M}_1 = (M_1, \omega_1)$, and let $T_1 = \{\, 2 \,\}$.
- $L_{T_1}(\mathcal{M}_1) = L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}$.

.

Example 1 (Cont.)

$\mathcal{M}_1 = (M_1, \omega_1)$, where

$$\omega_1(t) = \begin{cases} 1, & \text{if } t \text{ is an accept transition,} \\ & \text{or a rewrite transition: } ab \to c \text{ or } abb \to d, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\omega_1}^{M_1}(w) = \begin{cases} 1, & \text{if } w \in L_1 = \{\, a^n c b^n c, a^n d b^{2n} d \mid n \geq 0 \,\}, \\ 2, & \text{if } w \in L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}, \\ 0, & \text{otherwise.} \end{cases}$$

- Let $\mathcal{M}_1 = (M_1, \omega_1)$, and let $T_1 = \{\, 2 \,\}$.
- $L_{T_1}(\mathcal{M}_1) = L_2 = \{\, a^n b^n c, a^n b^{2n} d \mid n \geq 0 \,\}$.
- Recall: M_1 is an RW-automaton, and $L_2 \notin \mathcal{L}(\mathrm{RW})$ [Jancar et al., 1998].

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- For each $(m, n), (m', n') \in \overline{\mathbb{N}} \times \mathbb{N}$, the operations $\oplus$ and $\odot$ are described as follows

$$\begin{aligned}(m, n) \oplus (m', n') &= (\max(m, m'), n + n'),\\ (m, n) \odot (m', n') &= (m + m', n \cdot n').\end{aligned}$$

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- For each $(m, n), (m', n') \in \overline{\mathbb{N}} \times \mathbb{N}$, the operations $\oplus$ and $\odot$ are described as follows

$$\begin{aligned} (m, n) \oplus (m', n') &= (\max(m, m'), n + n'), \\ (m, n) \odot (m', n') &= (m + m', n \cdot n'). \end{aligned}$$

- Let $\mathbb{H}_{exp}$ be a family of subsets of the semiring $\overline{\mathbb{N}} \times \mathbb{N}$, and it is defined as follows

$$\mathbb{H}_{exp} = \{\{(n, m) \in \overline{\mathbb{N}} \times \mathbb{N} \mid m \geq 2^{n-1}\} \mid n \geq 1\}.$$

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- For each $(m, n), (m', n') \in \overline{\mathbb{N}} \times \mathbb{N}$, the operations $\oplus$ and $\odot$ are described as follows

$$\begin{aligned} (m, n) \oplus (m', n') &= (\max(m, m'), n + n'), \\ (m, n) \odot (m', n') &= (m + m', n \cdot n'). \end{aligned}$$

- Let $\mathbb{H}_{exp}$ be a family of subsets of the semiring $\overline{\mathbb{N}} \times \mathbb{N}$, and it is defined as follows

$$\mathbb{H}_{exp} = \{\{(n, m) \in \overline{\mathbb{N}} \times \mathbb{N} \mid m \geq 2^{n-1}\} \mid n \geq 1\}.$$

- Let $\varphi(\alpha)$ be the encoding of 3-SAT given by Jurdziński et al.(2004), where α is a Boolean formula in conjunctive normal form of degree 3.

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- For each $(m, n), (m', n') \in \overline{\mathbb{N}} \times \mathbb{N}$, the operations $\oplus$ and $\odot$ are described as follows

$$\begin{aligned}(m, n) \oplus (m', n') &= (\max(m, m'), n + n'),\\ (m, n) \odot (m', n') &= (m + m', n \cdot n').\end{aligned}$$

- Let $\mathbb{H}_{exp}$ be a family of subsets of the semiring $\overline{\mathbb{N}} \times \mathbb{N}$, and it is defined as follows

$$\mathbb{H}_{exp} = \{\{(n, m) \in \overline{\mathbb{N}} \times \mathbb{N} \mid m \geq 2^{n-1}\} \mid n \geq 1\}.$$

- Let $\varphi(\alpha)$ be the encoding of 3-SAT given by Jurdziński et al.(2004), where α is a Boolean formula in conjunctive normal form of degree 3.
- Let $N(\alpha)$ denote the number of satisfying assignments for the Boolean formula α.

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

- Let the semirings $\overline{\mathbb{N}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- For each $(m, n), (m', n') \in \overline{\mathbb{N}} \times \mathbb{N}$, the operations $\oplus$ and $\odot$ are described as follows

$$\begin{aligned}(m, n) \oplus (m', n') &= (\max(m, m'), n + n'),\\ (m, n) \odot (m', n') &= (m + m', n \cdot n').\end{aligned}$$

- Let $\mathbb{H}_{exp}$ be a family of subsets of the semiring $\overline{\mathbb{N}} \times \mathbb{N}$, and it is defined as follows

$$\mathbb{H}_{exp} = \{\{(n, m) \in \overline{\mathbb{N}} \times \mathbb{N} \mid m \geq 2^{n-1}\} \mid n \geq 1\}.$$

- Let $\varphi(\alpha)$ be the encoding of 3-SAT given by Jurdziński et al.(2004), where α is a Boolean formula in conjunctive normal form of degree 3.
- Let $N(\alpha)$ denote the number of satisfying assignments for the Boolean formula α.
- Let the language

$$L_{3SAT,half} = \{\varphi(\alpha) \mid \quad \begin{array}{l}\alpha \in \text{3-SAT}, N(\alpha) \geq 2^{n-1},\\ \text{where } \alpha \text{ contains the variables } x_1, x_2, \ldots, x_n\}.\end{array}$$

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

Theorem 6

$L_{3SAT,half} \in \mathcal{L}(\mathsf{RWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H}_{exp})$.

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

Theorem 6

$L_{3SAT,half} \in \mathcal{L}(\mathrm{RWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H}_{exp})$.

Recall

- The problem of determining the number of satisfying truth assignments for a Boolean formula is a #P-complete problem [Valiant, 1979].

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

Theorem 6

$L_{3SAT,half} \in \mathcal{L}(\mathrm{RWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H}_{exp})$.

Recall

- The problem of determining the number of satisfying truth assignments for a Boolean formula is a #P-complete problem [Valiant, 1979].
- The language class $\mathcal{L}(\mathrm{RWW})$ is reducible in linear time to the language class $\mathcal{L}(\mathrm{R})$ by using a further encoding technique [Otto, 2006].

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

Theorem 6

$L_{3SAT,half} \in \mathcal{L}(\mathrm{RWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H}_{exp})$.

Recall

- The problem of determining the number of satisfying truth assignments for a Boolean formula is a #P-complete problem [Valiant, 1979].
- The language class $\mathcal{L}(\mathrm{RWW})$ is reducible in linear time to the language class $\mathcal{L}(\mathrm{R})$ by using a further encoding technique [Otto, 2006].

Corollary 7

A weighted R*-automaton can compute a* #P*-complete function.*

On the Class of Languages Accepted Relative to the Semiring $\overline{\mathbb{N}} \times \mathbb{N}$

Theorem 6

$L_{3SAT,half} \in \mathcal{L}(\mathrm{RWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H}_{exp})$.

Recall

- The problem of determining the number of satisfying truth assignments for a Boolean formula is a #P-complete problem [Valiant, 1979].
- The language class $\mathcal{L}(\mathrm{RWW})$ is reducible in linear time to the language class $\mathcal{L}(\mathrm{R})$ by using a further encoding technique [Otto, 2006].

Corollary 7

A weighted R*-automaton can compute a* #P*-complete function.*

Theorem 8

The language class $\mathcal{L}(\mathrm{RRWW}, \overline{\mathbb{N}} \times \mathbb{N}, \mathbb{H})$ *is closed under the operation of reversal for each family* $\mathbb{H}$ *of subsets of* $\overline{\mathbb{N}} \times \mathbb{N}$.

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Definition 9

A weighted restarting automaton $\mathcal{M} = (M, \omega)$ of type wX is called a *regular-weighted restarting automaton* of type X (a $\mathrm{w_{REG}X}$-automaton for short), if M is a restarting automaton of type X, and ω is a weight function from the transitions of M into the semiring $(\mathbb{P}(\Delta^*), \cup, \cdot, \emptyset, \{\lambda\})$ such that the weight $\omega(t)$ of each transition t of M is a regular language over Δ, i.e., $\omega(t) \in \mathrm{REG}(\Delta)$.

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Definition 9

A weighted restarting automaton $\mathcal{M} = (M, \omega)$ of type wX is called a *regular-weighted restarting automaton* of type X (a $\mathsf{w_{REG}}$X-automaton for short), if M is a restarting automaton of type X, and ω is a weight function from the transitions of M into the semiring $(\mathbb{P}(\Delta^*), \cup, \cdot, \emptyset, \{\lambda\})$ such that the weight $\omega(t)$ of each transition t of M is a regular language over Δ, i.e., $\omega(t) \in \mathsf{REG}(\Delta)$.

Definition 10

Let $\mathcal{M} = (M, \omega)$ be a $\mathsf{w_{REG}}$X-automaton with input alphabet Σ, where ω maps the transitions of M to regular languages over Δ.

(a) For a set $T \in \mathbb{P}(\Delta^*)$, $\hat{L}_T(\mathcal{M}) = \{\, w \in L(M) \mid f^M_\omega(w) \cap T \neq \emptyset \,\}$ is the language *weakly* accepted by $\mathcal{M}$ relative to the set T, that is, a word $w \in \Sigma^*$ belongs to the language $\hat{L}_T(\mathcal{M})$ iff $w \in L(M)$ and $f^M_\omega(w)$ contains an element of T.

(b) Let the family $\mathbb{H} \subseteq \mathbb{P}(\Delta^*)$. Then

$$\hat{\mathcal{L}}(\mathsf{w_{REG}}\mathsf{X}, \mathsf{REG}(\Delta), \mathbb{H}) = \{\, \hat{L}_T(\mathcal{M}) \mid \mathcal{M} \text{ is a } \mathsf{w_{REG}}\mathsf{X}\text{-automaton and } T \in \mathbb{H}\}$$

is the class of languages that are weakly accepted by $\mathsf{w_{REG}}$X-automata relative to $\mathbb{H}$.

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Let σ be the mapping that is given through

(1) $\sigma(w) = w\#^n$, where $w \in \Sigma^*$ and $|w| = n$,

(2) $\sigma(L) = \{\, \sigma(w) \mid w \in L \,\}$ for a language $L \subseteq \Sigma^*$,

(3) $\sigma(\mathcal{L}) = \{\, \sigma(L) \mid L \in \mathcal{L} \,\}$ for a set of languages $\mathcal{L} \subseteq \mathbb{P}(\Sigma^*)$.

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Let σ be the mapping that is given through

(1) $\sigma(w) = w\#^n$, where $w \in \Sigma^*$ and $|w| = n$,

(2) $\sigma(L) = \{\, \sigma(w) \mid w \in L \,\}$ for a language $L \subseteq \Sigma^*$,

(3) $\sigma(\mathcal{L}) = \{\, \sigma(L) \mid L \in \mathcal{L} \,\}$ for a set of languages $\mathcal{L} \subseteq \mathbb{P}(\Sigma^*)$.

Theorem 11

$\sigma(\mathcal{L}(\mathsf{RRWW})) \subseteq \hat{\mathcal{L}}(\mathsf{w_{REG}RW}, \mathsf{REG}(\Delta), \mathsf{DCFL}(\Delta))$.

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Let σ be the mapping that is given through

(1) $\sigma(w) = w\#^n$, where $w \in \Sigma^*$ and $|w| = n$,

(2) $\sigma(L) = \{ \sigma(w) \mid w \in L \}$ for a language $L \subseteq \Sigma^*$,

(3) $\sigma(\mathcal{L}) = \{ \sigma(L) \mid L \in \mathcal{L} \}$ for a set of languages $\mathcal{L} \subseteq \mathbb{P}(\Sigma^*)$.

Theorem 11

$\sigma(\mathcal{L}(\mathsf{RRWW})) \subseteq \hat{\mathcal{L}}(\mathsf{w_{REG}RW}, \mathsf{REG}(\Delta), \mathsf{DCFL}(\Delta))$.

Proof Sketch:

- Recall: meta-instruction $(¢ \cdot E_1, u \to v, E_2 \cdot \$)$, where E_1 and E_2 are regular languages.
- $$T = \{ \beta_1^R[u_1, v_1]\alpha_1^R \& \alpha_1 v_1 \beta_1 \#^{|u_1|-|v_1|} \beta_2^R[u_2, v_2]\alpha_2^R \& \alpha_2 v_2 \beta_2 \#^{|u_2|-|v_2|} \ldots \beta_n^R[u_n, v_n]\alpha_n^R \& \alpha_n v_n \beta_n \#^{|u_n|-|v_n|} @ \mid \alpha_1, u_1, \beta_1 \in \Sigma^*, \alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n, u_2, \ldots, u_n, v_1, \ldots, v_n \in \Gamma^* \}.$$

On the Expressive Power of Regular-Weighted Restarting Automata with a Weak Acceptance Condition

Let σ be the mapping that is given through

(1) $\sigma(w) = w\#^n$, where $w \in \Sigma^*$ and $|w| = n$,

(2) $\sigma(L) = \{ \sigma(w) \mid w \in L \}$ for a language $L \subseteq \Sigma^*$,

(3) $\sigma(\mathcal{L}) = \{ \sigma(L) \mid L \in \mathcal{L} \}$ for a set of languages $\mathcal{L} \subseteq \mathbb{P}(\Sigma^*)$.

Theorem 11

$\sigma(\mathcal{L}(\mathrm{RRWW})) \subseteq \hat{\mathcal{L}}(\mathrm{w_{REG}RW}, \mathrm{REG}(\Delta), \mathrm{DCFL}(\Delta))$.

Proof Sketch:

- Recall: meta-instruction $(¢ \cdot E_1, u \to v, E_2 \cdot \$)$, where E_1 and E_2 are regular languages.
- $$T = \{ \beta_1^R[u_1, v_1]\alpha_1^R\&\alpha_1 v_1 \beta_1 \#^{|u_1|-|v_1|} \beta_2^R[u_2, v_2]\alpha_2^R\&\alpha_2 v_2 \beta_2 \#^{|u_2|-|v_2|} \ldots \beta_n^R[u_n, v_n]\alpha_n^R\&\alpha_n v_n \beta_n \#^{|u_n|-|v_n|} @ \mid \alpha_1, u_1, \beta_1 \in \Sigma^*, \alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n, u_2, \ldots, u_n, v_1, \ldots, v_n \in \Gamma^* \}.$$

Corollary 12

For $\mathrm{X} \in \{\lambda, \mathrm{W}, \mathrm{WW}\}$, $\sigma(\mathcal{L}(\mathrm{RRX})) \subseteq \hat{\mathcal{L}}(\mathrm{w_{REG}RX}, \mathrm{REG}(\Delta), \mathrm{DCFL}(\Delta))$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 13

A weighted restarting automaton $\mathcal{M} = (M, \omega)$ of type wX is called a *word-weighted restarting automaton* of type X (a $\mathrm{w_{word}X}$-automaton for short), if M is a restarting automaton of type X, and ω is a weight function from the transitions of M into the semiring $(\mathbb{P}_{\mathrm{fin}}(\Delta^*), \cup, \cdot, \emptyset, \{\lambda\})$ such that the weight of each transition t of M is of the form $\omega(t) = \{v\}$ for some $v \in \Delta^*$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 13

A weighted restarting automaton $\mathcal{M} = (M, \omega)$ of type wX is called a *word-weighted restarting automaton* of type X (a $\mathrm{w_{word}}$X-automaton for short), if M is a restarting automaton of type X, and ω is a weight function from the transitions of M into the semiring $(\mathbb{P}_{\text{fin}}(\Delta^*), \cup, \cdot, \emptyset, \{\lambda\})$ such that the weight of each transition t of M is of the form $\omega(t) = \{v\}$ for some $v \in \Delta^*$.

Definition 14

Let $\mathcal{M} = (M, \omega)$ be a word-weighted X-automaton with input alphabet Σ, where ω maps the transitions of M to singleton sets over Δ.

(a) For a set $T \subseteq \mathbb{P}_{\text{fin}}(\Delta^*)$, $L_T(\mathcal{M}) = \{\, w \in L(M) \mid f^M_\omega(w) \in T \,\}$ is the language *strongly accepted by* $\mathcal{M}$ *relative to* the set T, that is, a word $w \in \Sigma^*$ belongs to the language $L_T(\mathcal{M})$ iff $w \in L(M)$ and $f^M_\omega(w)$ is an element of T.

(b) Let $\mathbb{H}$ be a family of the subsets of $\mathbb{P}_{\text{fin}}(\Delta^*)$. Then

$$\mathcal{L}(\mathrm{w_{word}}\mathrm{X}, \mathbb{P}_{\text{fin}}(\Delta^*), \mathbb{H}) = \{\, L_T(\mathcal{M}) \mid \mathcal{M} \text{ is a } \mathrm{w_{word}}\mathrm{X}\text{-automaton and } T \in \mathbb{H}\}$$

is the class of languages that are strongly accepted by $\mathrm{w_{word}}$X-automata relative to $\mathbb{H}$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathrm{REG}(\Delta)$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathrm{REG}(\Delta)$.

Theorem 16

For all $\mathsf{X} \in \{\mathsf{R}, \mathsf{RR}, \mathsf{RW}, \mathsf{RRW}, \mathsf{RWW}, \mathsf{RRWW}\}$, $\mathcal{L}(\text{nf-}\mathsf{X}) \subseteq \mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathrm{REG}(\Delta)$.

Theorem 16

For all $\mathsf{X} \in \{\mathsf{R}, \mathsf{RR}, \mathsf{RW}, \mathsf{RRW}, \mathsf{RWW}, \mathsf{RRWW}\}$, $\mathcal{L}(\text{nf-}\mathsf{X}) \subseteq \mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

Observation

- $L_{copy} = \{\, uu \mid u \in \Sigma^* \,\}$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathrm{REG}(\Delta)$.

Theorem 16

For all $\mathrm{X} \in \{\mathrm{R}, \mathrm{RR}, \mathrm{RW}, \mathrm{RRW}, \mathrm{RWW}, \mathrm{RRWW}\}$, $\mathcal{L}(\text{nf-X}) \subseteq \mathcal{L}(\mathrm{w_{word}X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

Observation

- $L_{copy} = \{\, uu \mid u \in \Sigma^* \,\}$.
- $L_{copy} \in \mathcal{L}(\text{nf-RR})$?

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathrm{REG}(\Delta)$.

Theorem 16

For all $\mathrm{X} \in \{\mathrm{R}, \mathrm{RR}, \mathrm{RW}, \mathrm{RRW}, \mathrm{RWW}, \mathrm{RRWW}\}$, $\mathcal{L}(\text{nf-X}) \subseteq \mathcal{L}(\mathrm{w_{word}X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

Observation

- $L_{copy} = \{\, uu \mid u \in \Sigma^* \,\}$.
- $L_{copy} \in \mathcal{L}(\text{nf-RR})$?

Lemma 17

$L_{copy} \in \mathcal{L}(\mathrm{w_{word}R}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

Definition 15

Let $\mathbb{H}_{fin}^{reg(\Delta)}$ be the family of sets of finite languages over some finite alphabet Δ such that for each $T \in \mathbb{H}_{fin}^{reg(\Delta)}$, $\bigcup_{V \in T} V \in \mathsf{REG}(\Delta)$.

Theorem 16

For all $\mathsf{X} \in \{\mathsf{R}, \mathsf{RR}, \mathsf{RW}, \mathsf{RRW}, \mathsf{RWW}, \mathsf{RRWW}\}$, $\mathcal{L}(\text{nf-}\mathsf{X}) \subseteq \mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

Observation

- $L_{copy} = \{\, uu \mid u \in \Sigma^* \,\}$.
- $L_{copy} \in \mathcal{L}(\text{nf-}\mathsf{RR})$?

Lemma 17

$L_{copy} \in \mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{R}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

Conjecture

For $\mathsf{X} \in \{\mathsf{R}, \mathsf{RR}\}$, $\mathcal{L}(\text{nf-}\mathsf{X}) \subsetneq \mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{X}, \mathbb{P}_{\mathrm{fin}}(\Delta^*), \mathbb{H}_{fin}^{reg(\Delta)})$.

A Stronger Acceptance Condition for Word-Weighted and Regular-Weighted Restarting Automata

	$\mathcal{L}(\mathsf{w}_{\mathsf{REG}}\mathsf{X}, \mathsf{REG}(\Delta), \mathbb{H})$	$\mathcal{L}(\mathsf{w}_{\mathsf{word}}\mathsf{X}, \mathbb{P}_{\text{fin}}(\Delta^*), \mathbb{H})$
Intersection	$\surd$	$\surd$
Union	$\surd$	$\surd$
Concatenation	$\surd$	$\surd$
Complement	$\surd$	$\surd$
Reversal	**?**	**?**

Table: Closure properties.

Open Problems

1. Which language classes can be characterized by weighted restarting automata of various types?
2. What is the upper bound of the expressive power of weighted restarting automata of some strong types such as RWW and RRWW?
3. Are regular-weighted restarting automata strictly more expressive than word-weighted restarting automata?