Hierarchies of Language Families of Contextual Grammars

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- Focus is often on the decrease of descriptional or computational complexity when going from arbitrary regular languages to special ones.
- Here, the generative capacity of contextual grammars with special selection languages is considered.

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- introduced by Solomon Marcus in 1969,
- formal model for generating languages,
- starting with an initial finite set of words,
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Contexts equipped with conditions where they can be applied

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Definitions

Contextual grammar with selection in $\mathcal{F}$ is a construct

$$G = (V, \{(S_1, C_1), (S_2, C_2), \ldots, (S_n, C_n)\}, A)$$

where

- V is an alphabet,
- for $1 \leq i \leq n$, S_i is a language over V in $\mathcal{F}$ (selection language),
- for $1 \leq i \leq n$, C_i is a finite set of pairs (u, v) (contexts) with $u, v \in V^*$,
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Internal derivation step: $x \Longrightarrow_{\mathrm{in}} y$ if $x = x_1x_2x_3$ with $x_2 \in S_i$ for some $i \in \{1, \ldots, n\}$ and $y = x_1ux_2vx_3$ for some $(u, v) \in C_i$.

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Languages generated:

$$L_{\mathrm{ex}}(G) = \{z \mid x \Longrightarrow_{\mathrm{ex}}^{*} z \text{ for some } x \in A\},$$

$$L_{\mathrm{in}}(G) = \{z \mid x \Longrightarrow_{\mathrm{in}}^{*} z \text{ for some } x \in A\}$$

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$\mathcal{EC}(\mathcal{F})$ and $\mathcal{IC}(\mathcal{F})$: family of all languages generated externally or internally by contextual grammars with selection in $\mathcal{F}$.

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- *definite* if and only if it can be represented in the form $L = A \cup V^*B$ where A and B are finite subsets of V^*,
- *suffix-closed* (or *fully initial* or *multiple-entry* language) if and only if, for any words $x \in V^*$ and $y \in V^*$, the relation $xy \in L$ implies $y \in L$,

- *ordered* if and only if the language is accepted by some deterministic finite automaton $\mathcal{A} = (V, Z, z_0, F, \delta)$ where $(Z, \preceq)$ is a totally ordered set and, for any $a \in V$, the relation $z \preceq z'$ implies $\delta(z, a) \preceq \delta(z', a)$,

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- *non-counting* (or *star-free*) if and only if there is a natural number $k \geq 1$ such that, for any words $x \in V^*$, $y \in V^*$, and $z \in V^*$, it holds $xy^kz \in L$ if and only if $xy^{k+1}z \in L$,

- *power-separating* if and only if, there is a natural number $m \geq 1$ such that for any $x \in V^*$, either $J_x^m \cap L = \emptyset$ or $J_x^m \subseteq L$ where $J_x^m = \{ x^n \mid n \geq m \}$,

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number of non-terminals of a right-linear grammar bounded by n

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↑ number of states of a DFA bounded by n

Previous Work

Subregular family $\mathcal{F}$ – External derivation:

Jürgen Dassow: Contextual grammars with subregular choice.
Fundamenta Informaticae 64 (2005), 109–118.

$\mathcal{F} \in \{FIN, MON, COMB, NIL, DEF, COMM, SUF\}$
(finite, monoidal, combinational, nilpotent, definite, commutative, suffix-closed)

Jürgen Dassow, Florin Manea, Bianca Truthe: On external contextual grammars with subregular selection languages.
Theoretical Computer Science 449 (2012), 64–73 (DCFS 2011)

$\mathcal{F} \in \{CIRC, ORD, UF\}$, Selection with bounded resources
(circular, ordered, union-free)

Previous Work

Subregular family $\mathcal{F}$ – Internal derivation:

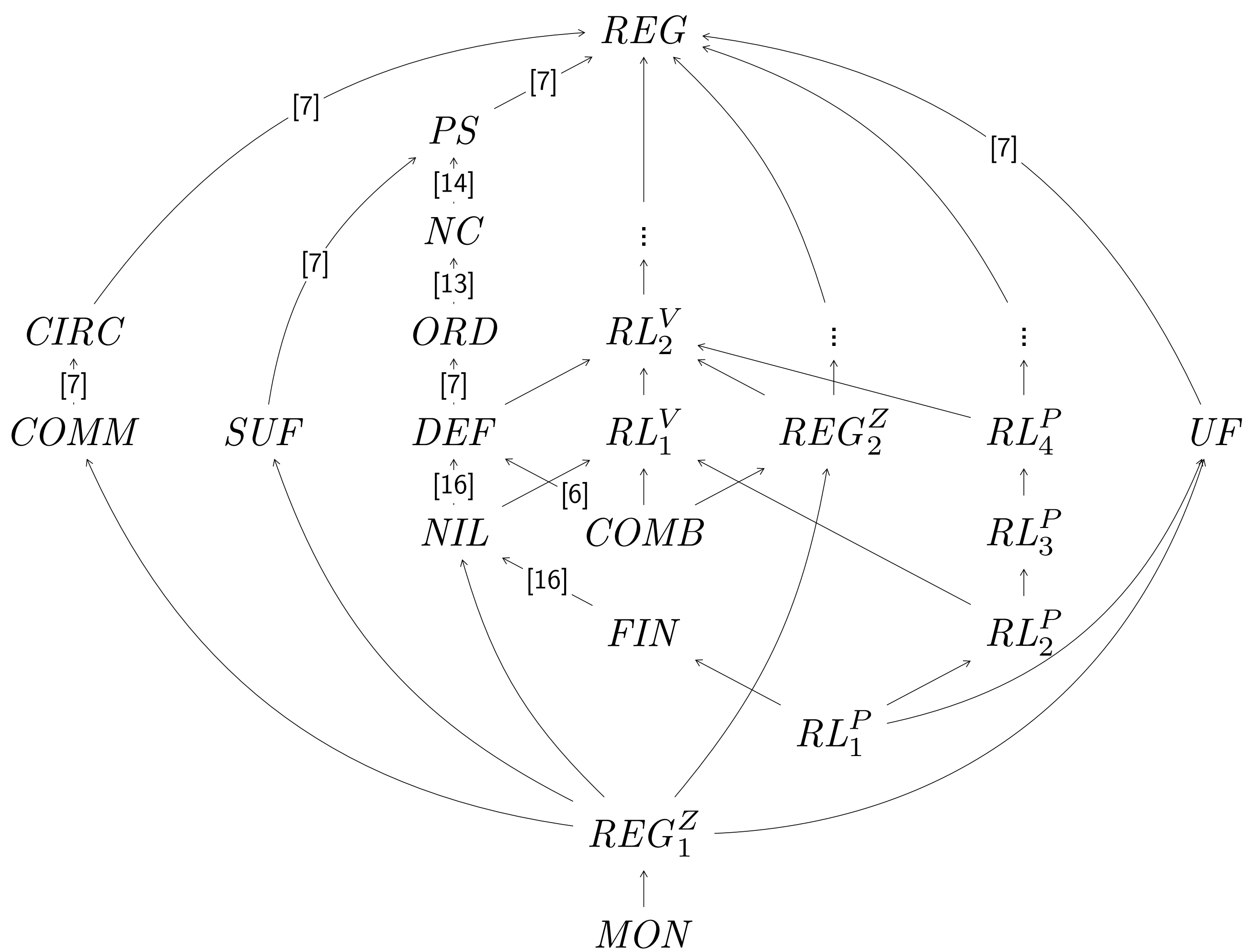
Jürgen Dassow, Florin Manea, Bianca Truthe: On subregular selection languages in internal contextual grammars.
Journal of Automata, Languages, and Combinatorics 17 (2012), 145–164 (DCFS 2012)

$\mathcal{F} \in \{FIN, MON, NIL, COMB, DEF, COMM, CIRC, SUF, UF\}$,
(finite, monoidal, nilpotent, combinational, definite, commutative, circular, suffix-closed, union-free)
Selection with bounded resources

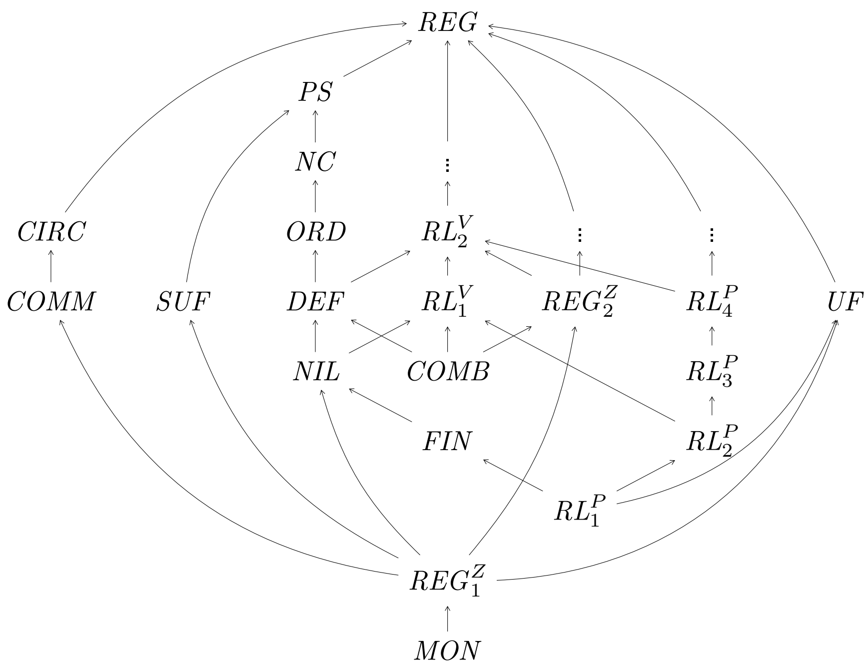
Bianca Truthe: A relation between definite and ordered finite automata.
NCMA 2014

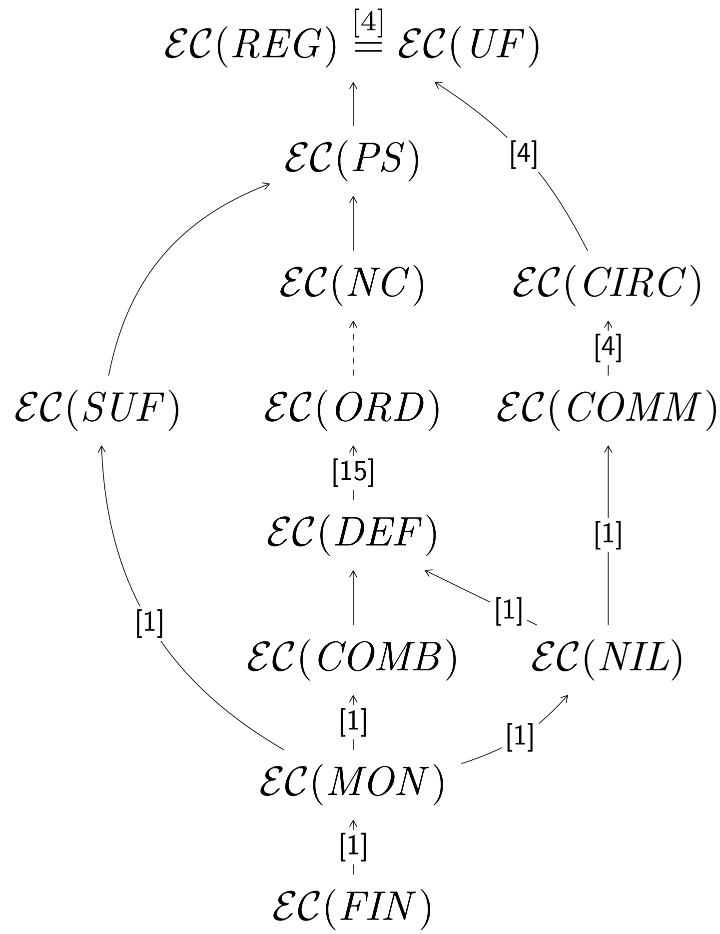
$\mathcal{F} \in \{DEF, ORD\}$

Hierarchy of Subregular Families



EC - Selection Languages of Special Structures





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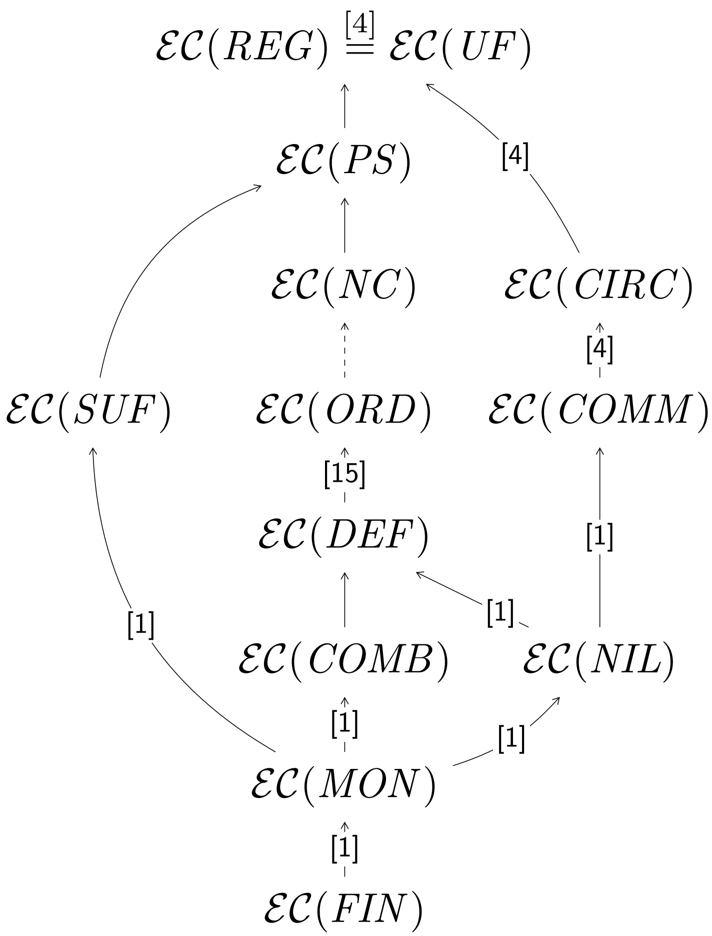

$$L_1 = \{aac\} \cup \{\, abc^n \mid n \geq 2 \,\} \cup \{\lambda\}$$
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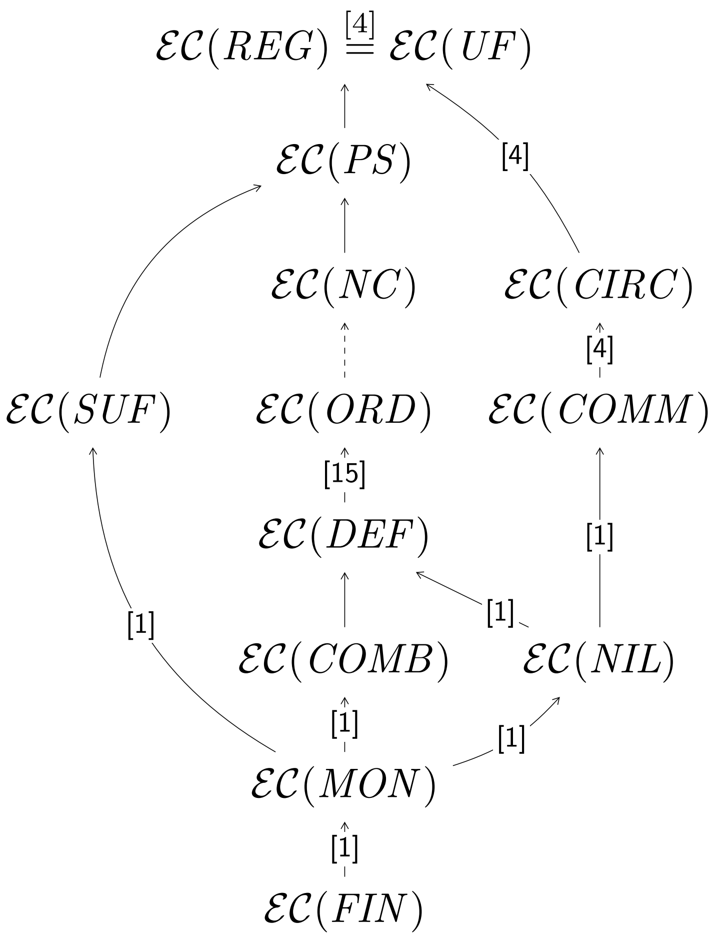

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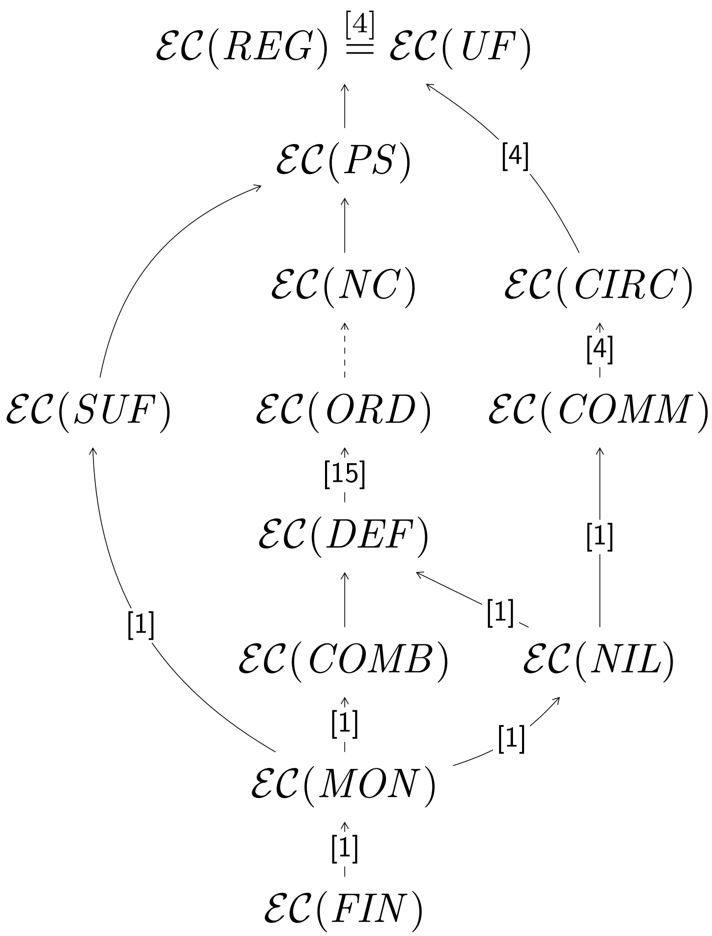

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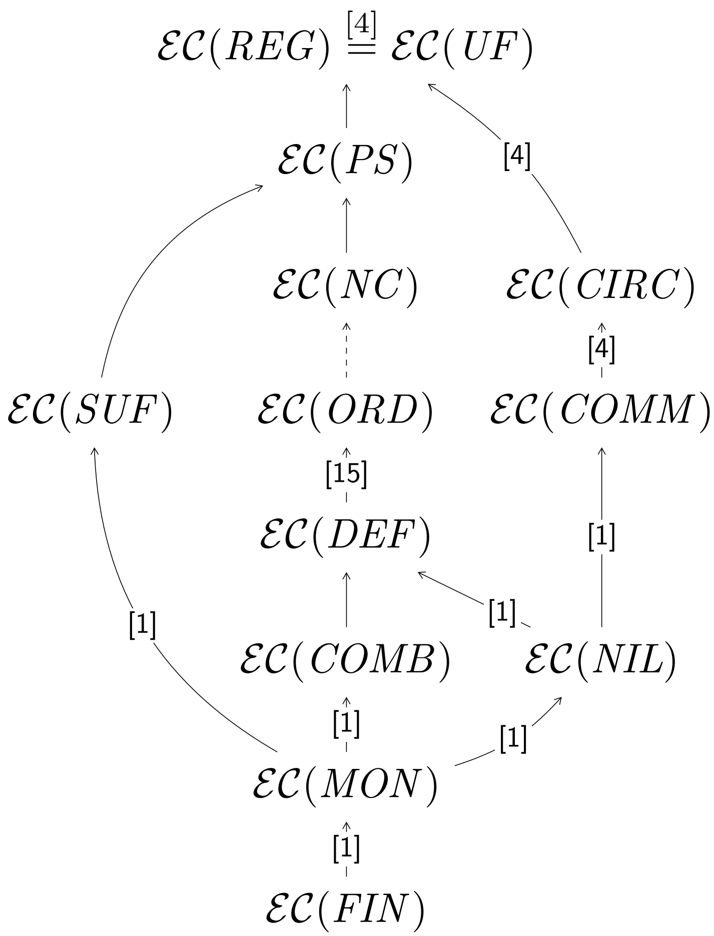

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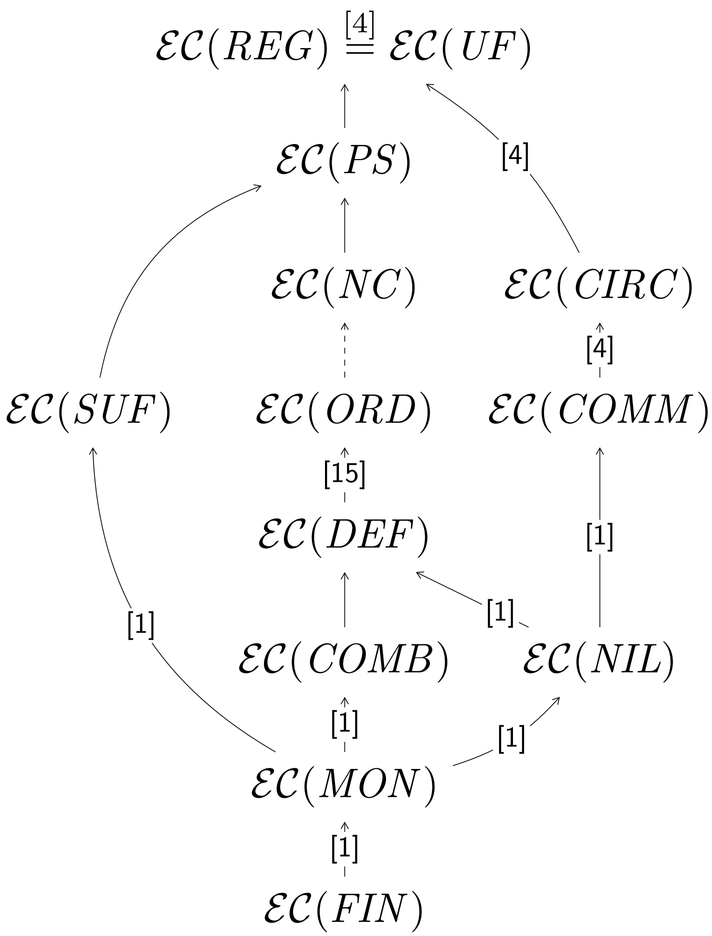
EC - Selection Languages of Special Structures



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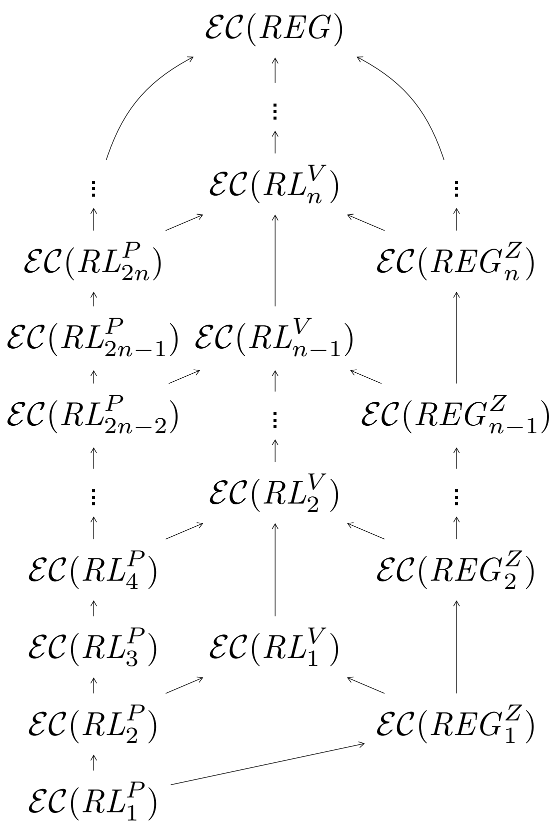
$$L_5 = \{\, ba^n \mid n \geq 1 \,\} \cup \{\, a^n b \mid n \geq 1 \,\} \cup \{\lambda\}$$
$$\in \mathcal{EC}(COMB) \setminus (\mathcal{EC}(CIRC) \cup \mathcal{EC}(SUF))$$

EC - Selection Languages with Bounded Resources



$$
\begin{aligned}
L_{n,1} &= \{a_1, a_2, \ldots, a_n\}^* \\
&\in \mathcal{EC}(RL_{n+1}^P) \cap \mathcal{EC}(RL_1^V) \cap \mathcal{EC}(REG_1^Z) \\
&\notin \mathcal{EC}(RL_n^P)
\end{aligned}
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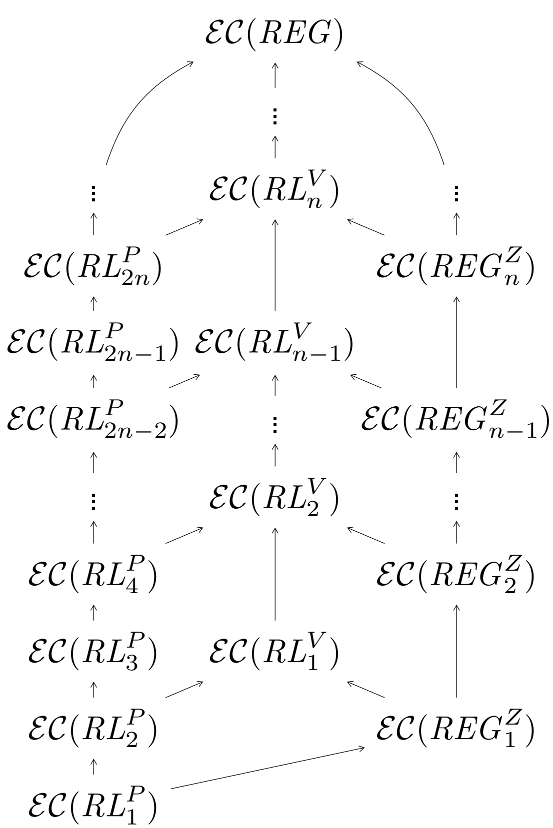

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$$\begin{aligned} L_{n,2} &= \{c\}((\{b\}^*\{a\})^n\{b\}^*)^+\{c\} \cup \{a,b\}^* \\ &\in \mathcal{EC}(RL_{n+1}^V) \cap \mathcal{EC}(REG_{n+1}^Z) \\ &\notin \mathcal{EC}(RL_n^V) \cup \mathcal{EC}(REG_n^Z) \end{aligned}$$

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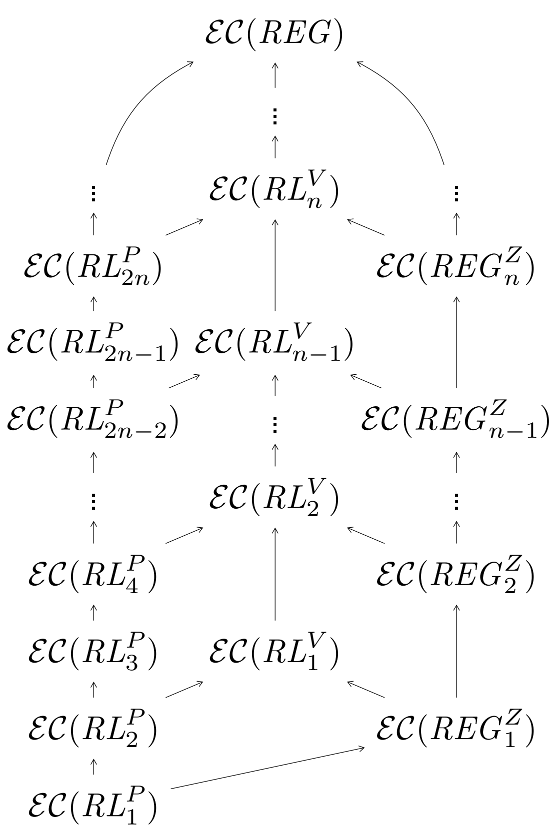

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$$\in \mathcal{EC}(RL_{2n+1}^P) \setminus \mathcal{EC}(RL_n^V)$$
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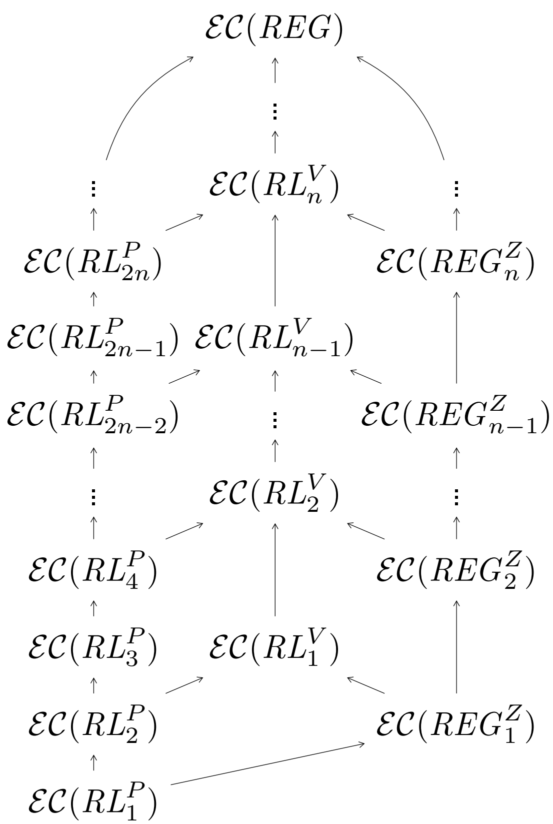
EC - Selection Languages with Bounded Resources



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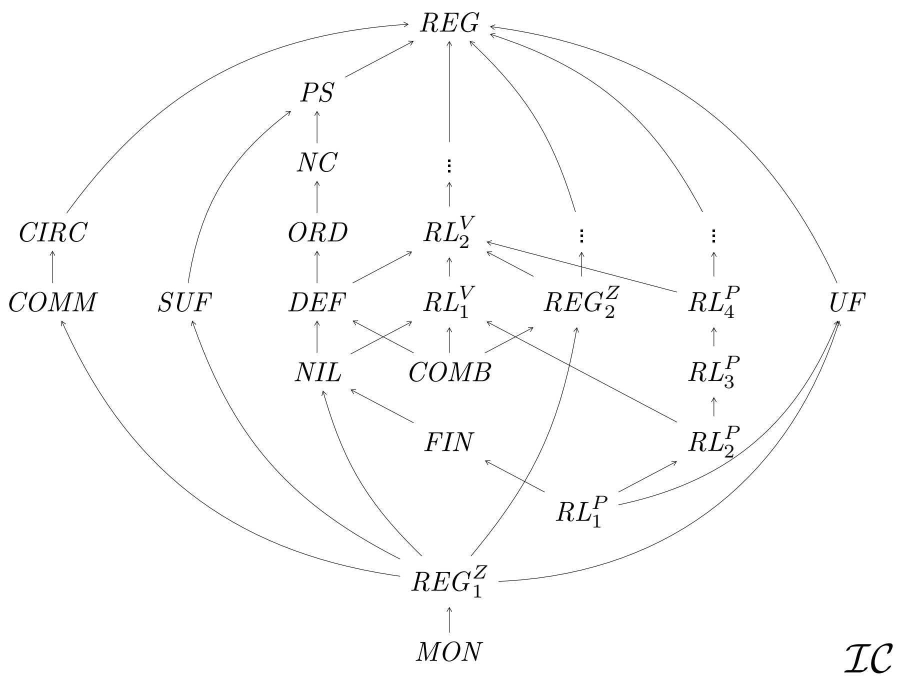
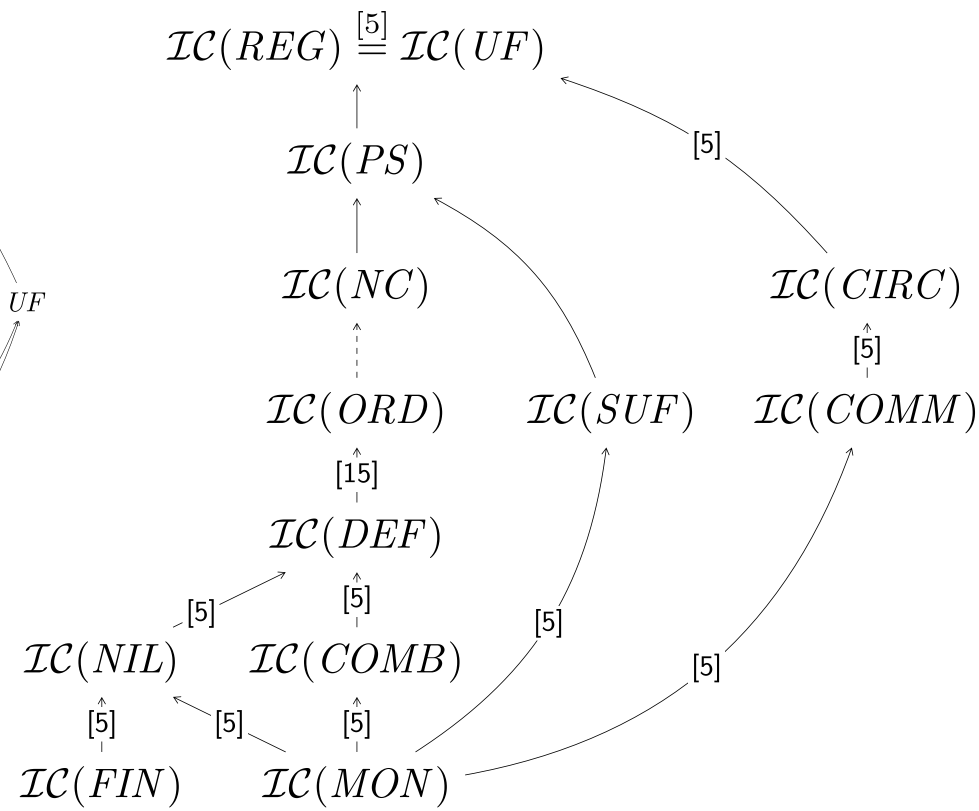
(with $V_n = \{a_1, a_2, \ldots, a_n, b\}$)

$$L_{n,4} = \{b\}\{a^{n+1}\}^*\{b\} \cup \{a\}^* \in (\mathcal{EC}(RL_2^P) \cap \mathcal{EC}(RL_1^V)) \setminus \mathcal{EC}(REG_n^Z)$$

IC - Selection Languages of Special Structures





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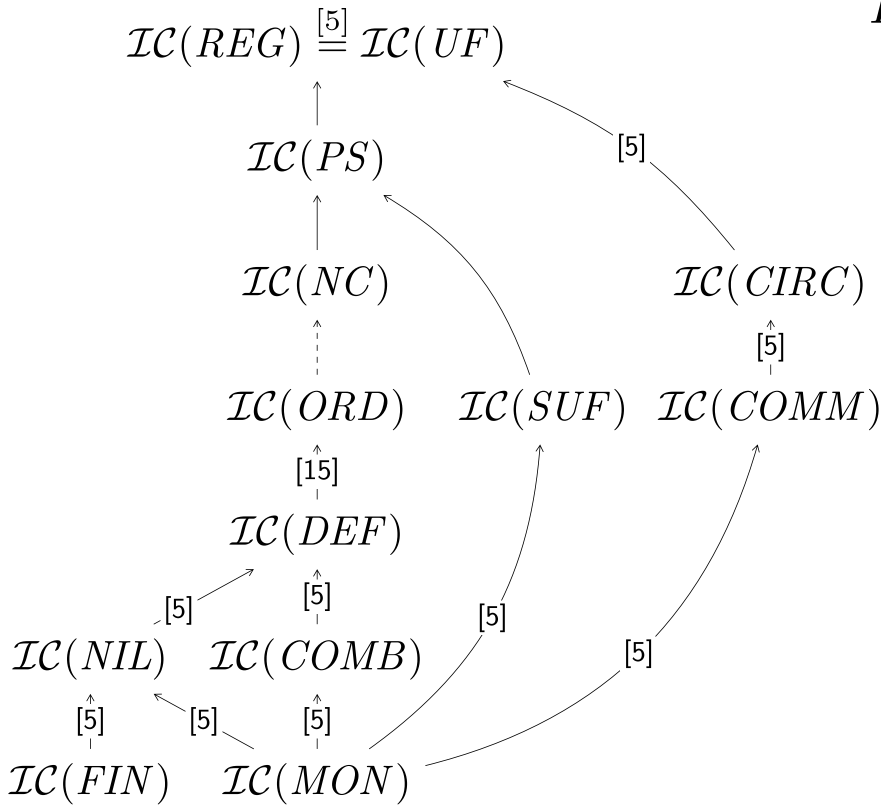
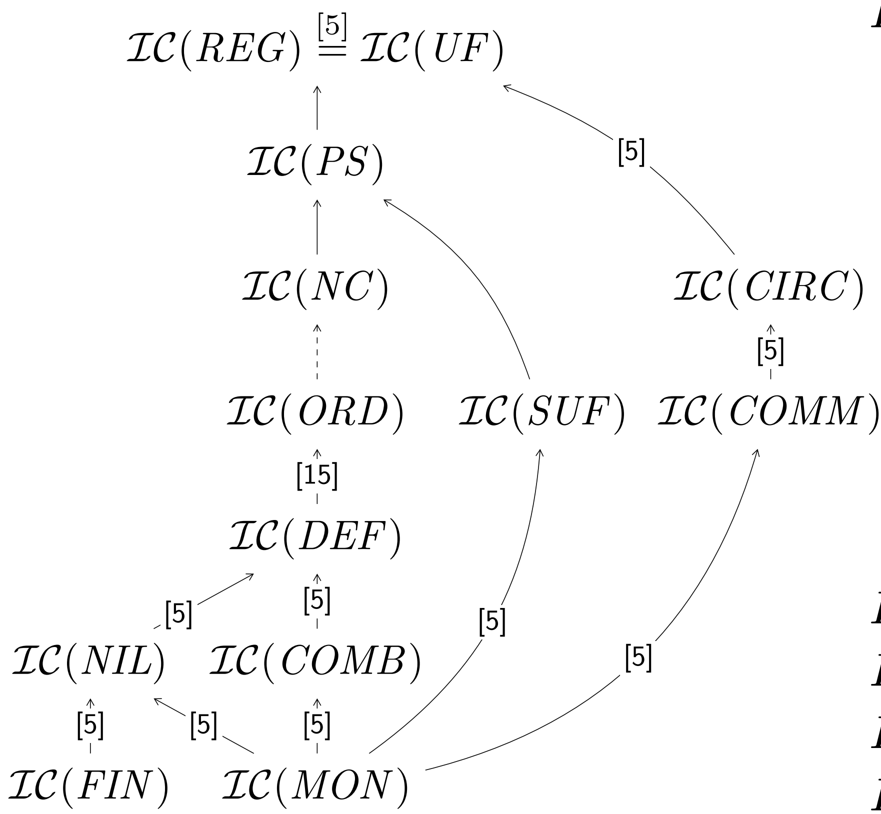


$$
\begin{aligned}
L_1 = &\{\, c^n a c^m b c^{n+m} \mid n \geq 0, m \geq 0 \,\} \\
&\cup \{\, c^n b c^n a \mid n \geq 0 \,\} \\
&\in \mathcal{IC}(FIN) \cap \mathcal{IC}(COMB) \\
&\notin \mathcal{IC}(SUF) \cup \mathcal{IC}(CIRC)
\end{aligned}
$$

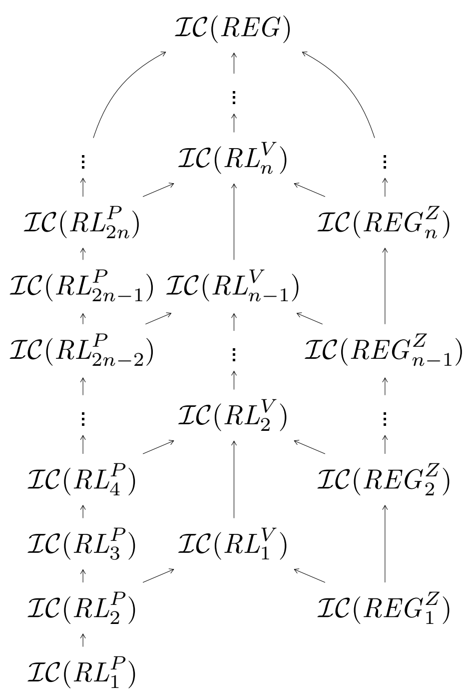
IC - Selection Languages of Special Structures



$$L_1 = \{\, c^n a c^m b c^{n+m} \mid n \geq 0, m \geq 0 \,\} \cup \{\, c^n b c^n a \mid n \geq 0 \,\}$$
$$\in \mathcal{IC}(FIN) \cap \mathcal{IC}(COMB)$$
$$\notin \mathcal{IC}(SUF) \cup \mathcal{IC}(CIRC)$$

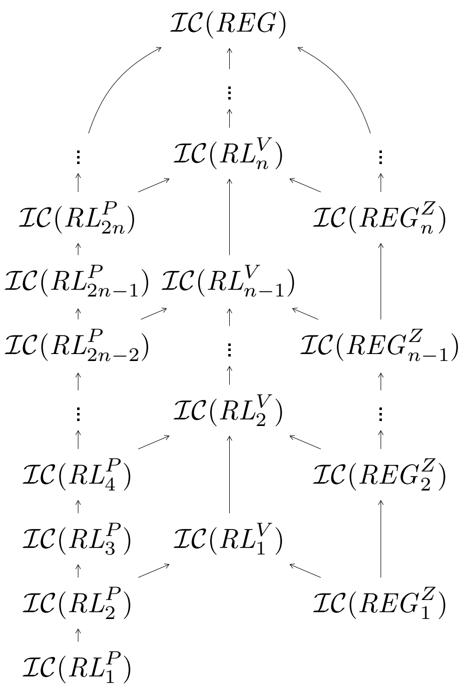
$$V = \{a, b, c, d, e\}$$
$$S_1 = \{a, b\}^*\{c\}\{a, b\}^*$$
$$C_1 = \{(ab, ab)\}$$
$$S_2 = \{aa\}^+$$
$$C_2 = \{(d, e)\}$$
$$L_2 = L((V, \{(S_1, C_1), (S_2, C_2)\}, \{c\}))$$
$$L_3 = L((V, \{(S_1, C_1), (\{b\}S_2\{b\}, C_2)\}, \{c\}))$$
$$L_2 \in \mathcal{IC}(COMM) \setminus \mathcal{IC}(PS)$$
$$L_3 \in \mathcal{IC}(PS) \setminus \mathcal{IC}(NC)$$

IC - Selection Languages with Bounded Resources



$$
\begin{aligned}
L_{n,1} &= \{\, a^{p_1}ba^{p_2}b\cdots a^{p_n}ba^{p_1}ba^{p_2}b\cdots a^{p_n} \mid p_i \geq 1, 1 \leq i \leq n \,\} \\
&\in \mathcal{IC}(RL_{2n-1}^P) \cap \mathcal{IC}(RL_n^V) \cap \mathcal{IC}(REG_{n+1}^Z) \\
&\notin \mathcal{IC}(RL_{2n-2}^P) \cup \mathcal{IC}(RL_{n-1}^V) \cap \mathcal{IC}(REG_n^Z)
\end{aligned}
$$

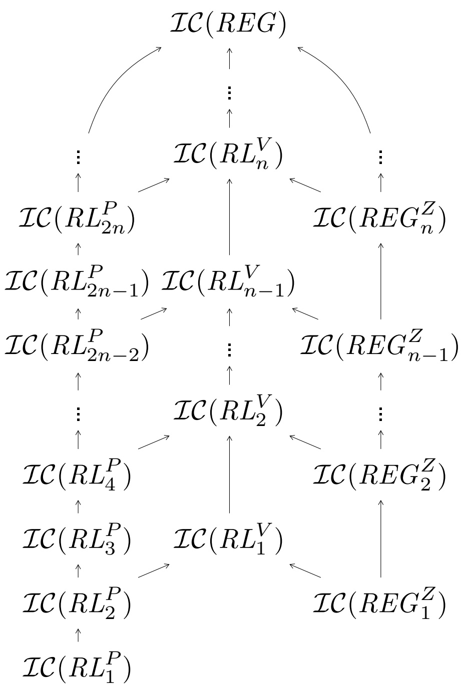
IC - Selection Languages with Bounded Resources



$$L_{n,1} = \{\ a^{p_1}ba^{p_2}b\cdots a^{p_n}ba^{p_1}ba^{p_2}b\cdots a^{p_n} \mid p_i \geq 1, 1 \leq i \leq n\ \}$$
$$\in \mathcal{IC}(RL_{2n-1}^P) \cap \mathcal{IC}(RL_n^V) \cap \mathcal{IC}(REG_{n+1}^Z)$$
$$\notin \mathcal{IC}(RL_{2n-2}^P) \cup \mathcal{IC}(RL_{n-1}^V) \cap \mathcal{IC}(REG_n^Z)$$

$$X_n = \{x_1, \ldots, x_n\}, X \in \{A, B, C, D\}$$
$$V_n = A_n \cup B_n \cup C_n \cup D_n$$
$$P_n = \{\ (a_i, c_j) \mid 1 \leq i \leq n, 1 \leq j \leq n\ \}$$
$$Q_n = \{\ (b_i, d_j) \mid 1 \leq i \leq n, 1 \leq j \leq n\ \}$$
$$L_{n,2} = L((V_n, \{(B_n^*, P_n), (C_n^*, Q_n)\},$$
$$\{\ a_{i_a}b_{i_b}c_{i_c}d_{i_d} \mid 1 \leq i_x \leq n, x \in \{a, b, c, d\}\ \}))$$
$$\in (\mathcal{IC}(REG_1^Z) \cap \mathcal{IC}(RL_1^V) \cap \mathcal{IC}(RL_{n+1}^P)) \setminus \mathcal{IC}(RL_n^P)$$

IC - Selection Languages with Bounded Resources



$$L_{n,1} = \{\ a^{p_1}ba^{p_2}b\cdots a^{p_n}ba^{p_1}ba^{p_2}b\cdots a^{p_n} \mid p_i \geq 1, 1 \leq i \leq n\ \}$$
$$\in \mathcal{IC}(RL_{2n-1}^P) \cap \mathcal{IC}(RL_n^V) \cap \mathcal{IC}(REG_{n+1}^Z)$$
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$$\{\ a_{i_a}b_{i_b}c_{i_c}d_{i_d} \mid 1 \leq i_x \leq n, x \in \{a, b, c, d\}\ \}))$$
$$\in (\mathcal{IC}(REG_1^Z) \cap \mathcal{IC}(RL_1^V) \cap \mathcal{IC}(RL_{n+1}^P)) \setminus \mathcal{IC}(RL_n^P)$$

$$V_n = \{a_1, \ldots, a_n\}$$
$$L_{n,3} = \{a_1a_2\cdots a_n\}^+ \cup V_n^{n-1}$$
$$\in (\mathcal{IC}(RL_1^P) \cap \mathcal{IC}(RL_1^V) \cap \mathcal{IC}(REG_{n+1}^Z)) \setminus \mathcal{IC}(REG_n^Z)$$

Future Work

- Open questions: $\mathcal{EC}(ORD) \subset \mathcal{EC}(NC)$? $\mathcal{IC}(ORD) \subset \mathcal{IC}(NC)$?

  $\mathcal{IC}(SUF) \subset \mathcal{IC}(ORD)$? $\mathcal{IC}(SUF) \subset \mathcal{IC}(NC)$?

- Inclusion relations and incomparabilities between classes based on structural properties and classes based on resources
- Other subregular families for the selection languages
- Further properties of the generated language families