Quantum Automata for Online Minimization Problems

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- Definitions.

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- Results.

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- Results.
- Main Ideas.

Online Minimization Problem

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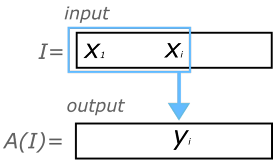

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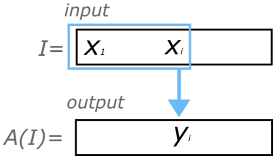

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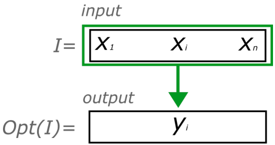

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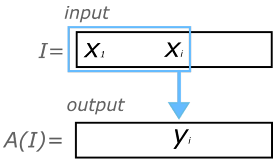

Competitive Ratio

- $Opt(I)$ is optimal offline solution;
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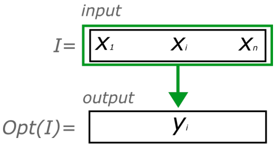

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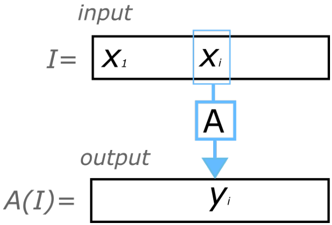

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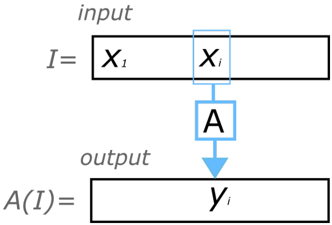

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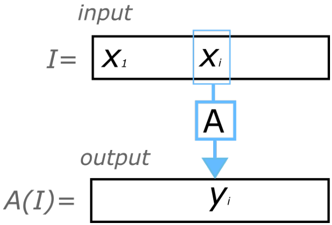
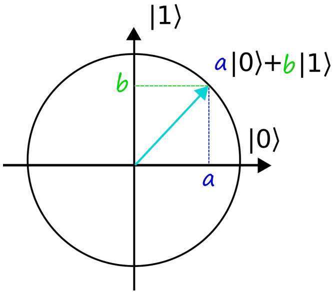


- J. Boyar, S. Irani, and K. Larsen, 2009;
- Y. Giannakopoulos and E. Koutsoupias, 2015;
- J. Boyar, K. Larsen, and A. Maiti, 2015.

Quantum Bit

Quantum bit

- State: $a|0\rangle + b|1\rangle$;



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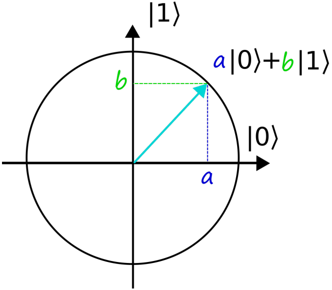

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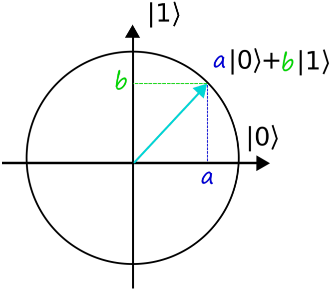

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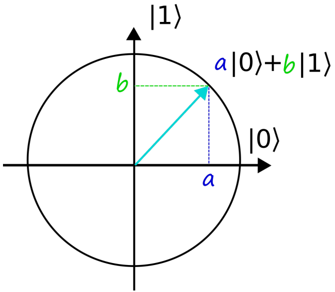

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Results on Sublinear Size

Theorems 3.1,3.4,3.5

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Results on Polynomial Size

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For special online minimization problem (n, w, r)-PNEH, for $w > r$ we have:

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$I=$

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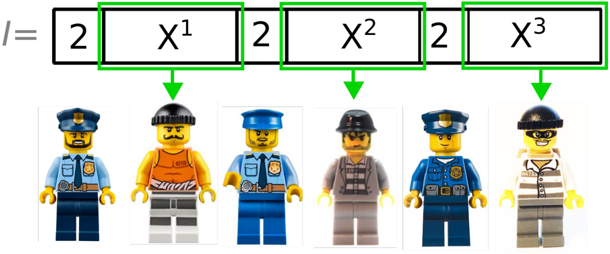

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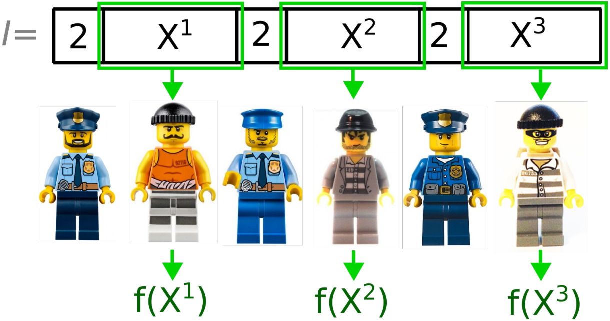

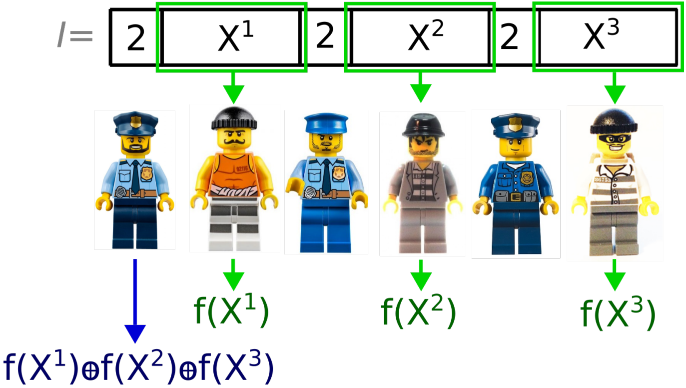
I=
2
X1
2
X2
2
X3
f(X1)
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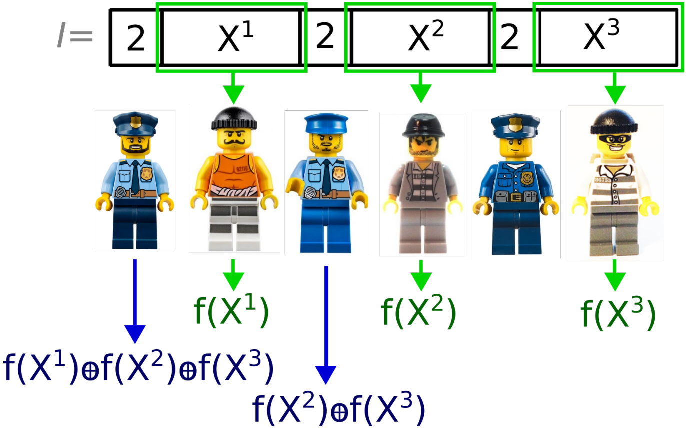

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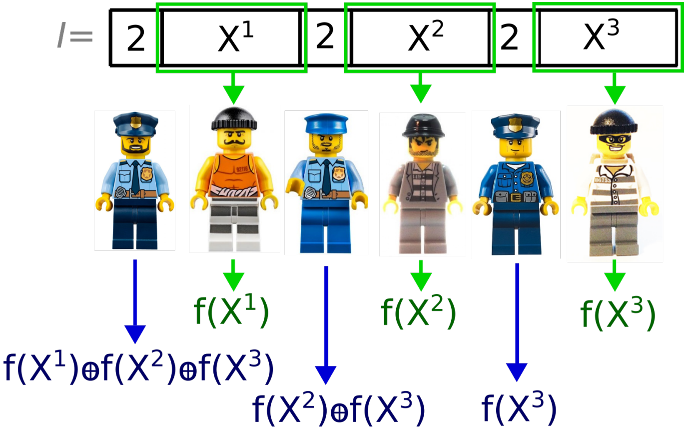

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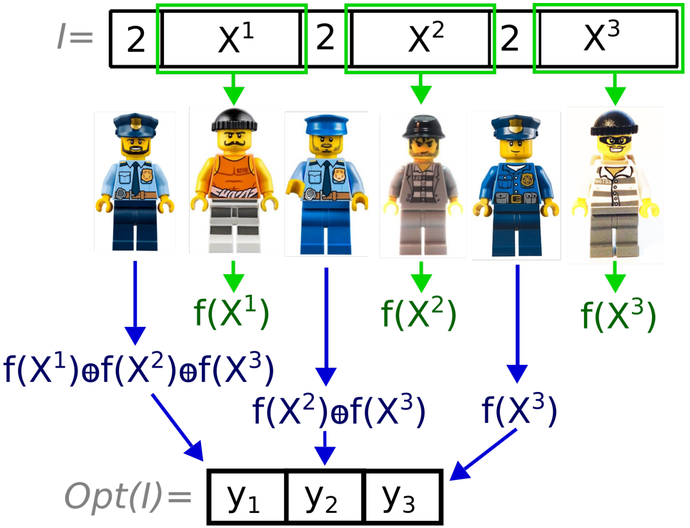

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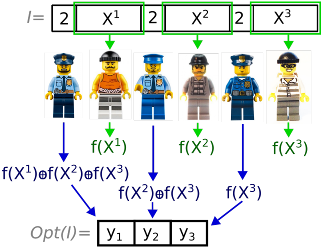

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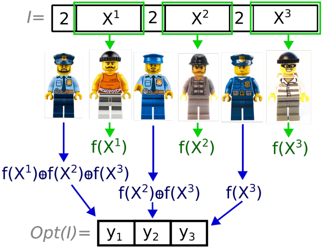

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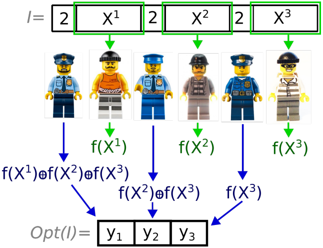

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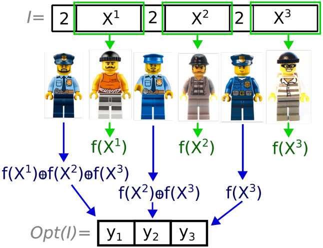
(n, k, w, r)-PNH: $f(X) = PartialMOD_k(X)$



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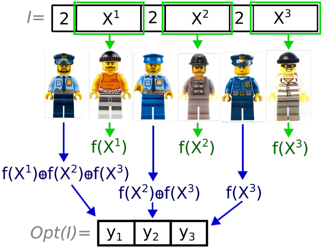

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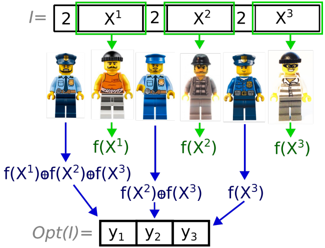

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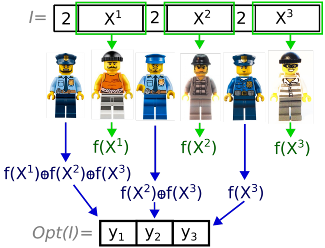

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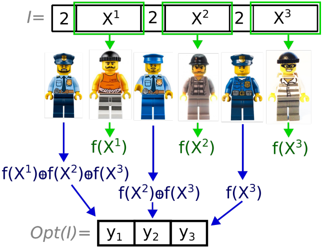
(n, w, r)-PNEH: $f(X) = EQ(X)$

- let $X = \sigma_1, \dots, \sigma_m, 3, \gamma_1, \dots, \gamma_t$;



Definitions of (n, k, w, r)-PNH and (n, w, r)-PNEH

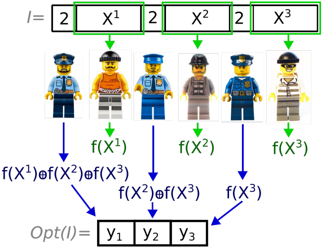
- $r < w$;
- $cost(I, O) = r$, if $y_1 = f(X_1) \oplus f(X_2) \oplus f(X_3)$, $y_2 = f(X_2) \oplus f(X_3)$ and $y_3 = f(X_3)$;
- $cost(I, O) = w$, otherwise.

(n, k, w, r)-PNH: $f(X) = PartialMOD_k(X)$

- Let $\#_1(X) = v \cdot 2^k$;
- $PartialMOD_k(X) = v \ mod \ 2$.

(n, w, r)-PNEH: $f(X) = EQ(X)$

- let $X = \sigma_1, \dots, \sigma_m, 3, \gamma_1, \dots, \gamma_t$;
- let $bin(\sigma_1, \dots, \sigma_m) = \sum_{i=1}^{m} 2^{i-1}\sigma_i$;



Definitions of (n,k,w,r)-PNH and (n,w,r)-PNEH

- $r < w$;
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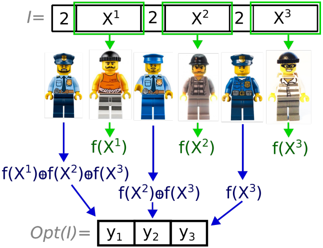
(n,k,w,r)-PNH: $f(X) = PartialMOD_k(X)$

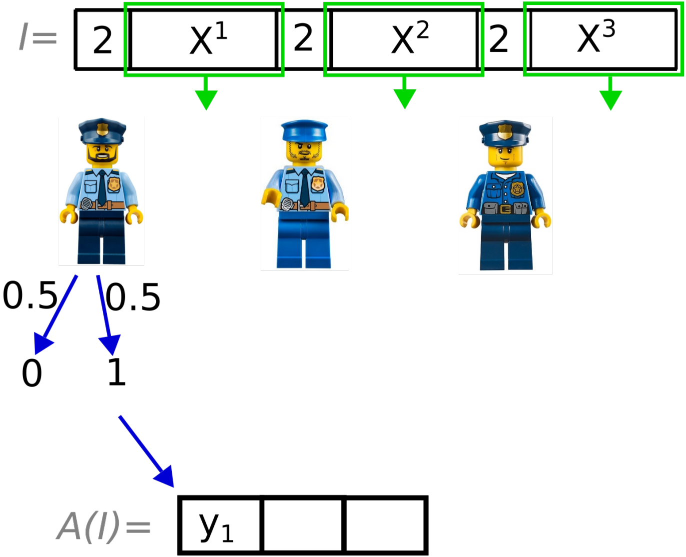
- Let $\#_1(X) = v \cdot 2^k$;
- $PartialMOD_k(X) = v \mod 2$.

(n,w,r)-PNEH: $f(X) = EQ(X)$

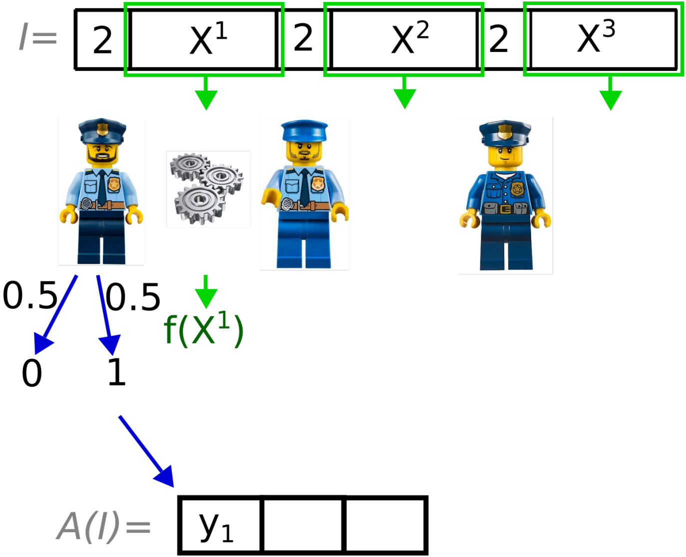
- let $X = \sigma_1,\dots,\sigma_m,3,\gamma_1,\dots,\gamma_t$;
- let $bin(\sigma_1,\dots,\sigma_m) = \sum_{i=1}^{m} 2^{i-1}\sigma_i$;
- $EQ(X) = 1$, iff $bin(\sigma_1,\dots,\sigma_m) = bin(\gamma_1,\dots,\gamma_t)$.



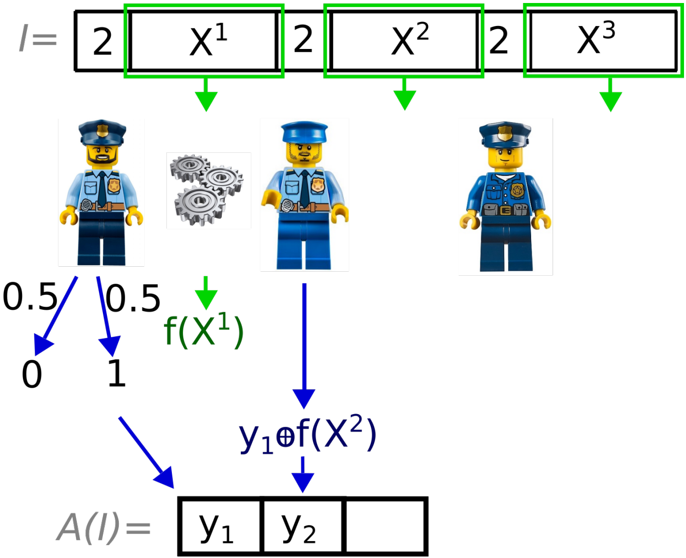
Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH



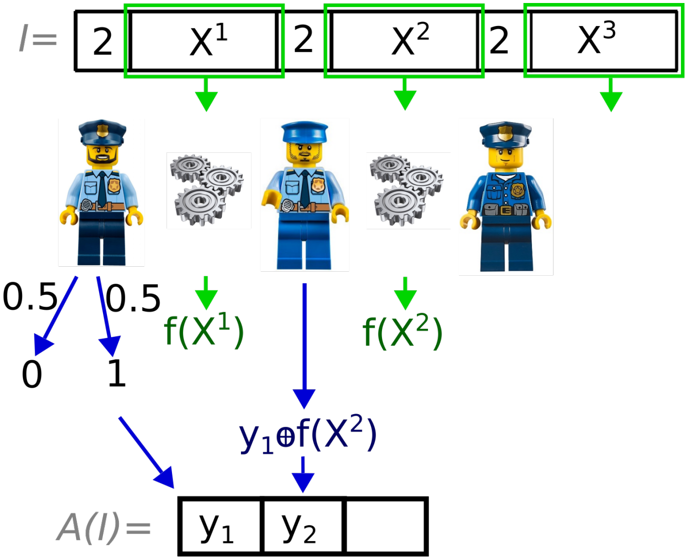
Quantum Automata for (n,k,w,r)-PNH and (n,w,r)-PNEH



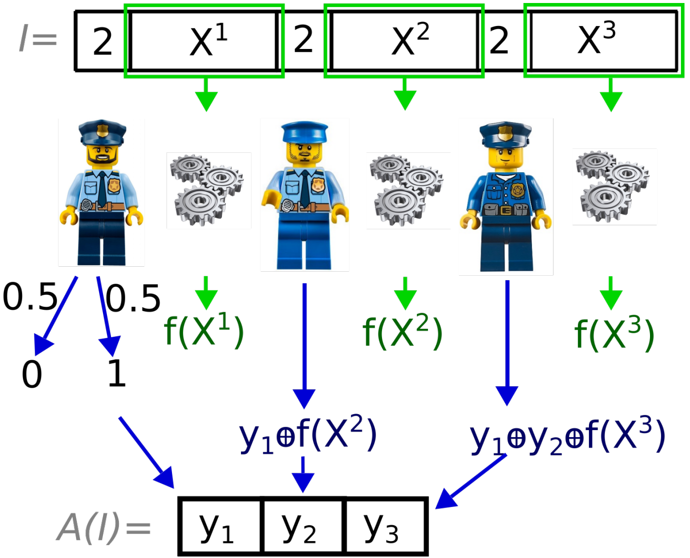
Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH



Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH



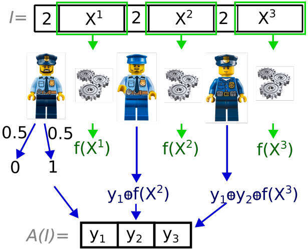
Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH



Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH

A. Ambainis and A. Yakaryılmaz, 2012: *PartialMOD$_k$*

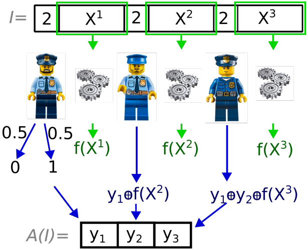
- There is 1 qubit exact quantum automaton.



Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH

A. Ambainis and A. Yakaryılmaz, 2012: *PartialMOD$_k$*

- There is 1 qubit exact quantum automaton.
- No deterministic or probabilistic automaton with less than 2^k states.



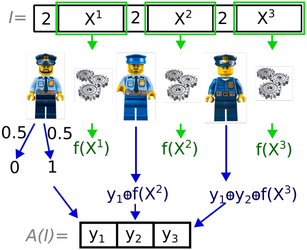
Quantum Automata for (n, k, w, r)-PNH and (n, w, r)-PNEH

A. Ambainis and A. Yakaryılmaz, 2012: *PartialMOD$_k$*

- There is 1 qubit exact quantum automaton.
- No deterministic or probabilistic automaton with less than 2^k states.

A. Ambainis and R. Freivalds, 1998; F. Ablayev and A. Vasilyev, 2009: *EQ*

- There is $n^{O(1)}$ states quantum automaton with one side error ε.
- No deterministic automaton with $2^{o(n)}$ states.



Thank you for your attention!

Děkuji!