Regular Grammars for Array Languages

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God completes the favors of wisdom, from the city of Trier

In this sign you will conquer

NCMA, Prague, Czech Republic, August 17-18, 2017

Map

- Regular Matrix Grammars
- (Regular : Regular) Array Grammars
- Isometric Regular Array Grammars
- Relation b/w language families

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Regular Matrix Grammars

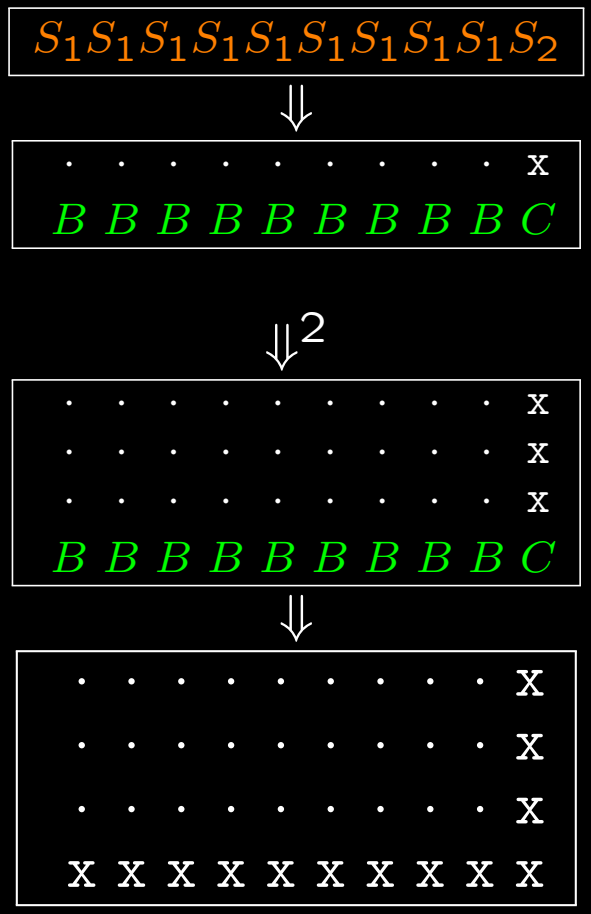
A *two-dimensional right-linear grammar (2RLG)* [G & R 97] is defined by a 7-tuple $G = (V_h, V_v, \Sigma_I, \Sigma, S, R_h, R_v)$, where:

- V_h is a finite set of *horizontal nonterminals*;
  V_v is a finite set of *vertical nonterminals*;
- $\Sigma_I \subseteq V_v$ is a finite set of *intermediates*;
  Σ is a finite set of *terminals*;
- $S \in V_h$ is a *starting symbol*;
- R_h is a finite set of *horizontal rules* of the form $V \to AV'$ or $V \to A$, where $V, V' \in V_h$ and $A \in \Sigma_I$;
- R_v is a finite set of *vertical rules* of the form $W \to aW'$ or $W \to a$, where $W, W' \in V_v$ and $a \in \Sigma$.

Original definition of a 2RLG (under the name of regular matrix grammar (RMG)) and properties of the corresponding class of picture languages, traditionally called *regular matrix languages* (RML), [Siromoney et al 72]. We use $\mathcal{L}_\Sigma(\mathrm{RMG})$ for convenience.

Example

$S \Rightarrow S_1A \Rightarrow^8 S_1S_1S_1S_1S_1S_1S_1S_1S_1A \Rightarrow$



A *(RMG)* $G = (V_h, V_v, \Sigma_I, \Sigma, S, R_h, R_v)$, where:

- $V_h = \{S, A\}$; $V_v = \{S_1, B, S_2, C\}$;
- $\Sigma_I = \{S_1, S_2\} \subseteq V_v$; $(V_h \cap \Sigma_I = \emptyset)$
  $\Sigma = \{\cdot, \mathtt{x}\}$; $S \in V_h$;
- $R_h = \{S \to S_1A,\ A \to S_1A,$
  $A \to S_2\}$;
- $R_v = \{S_1 \to \cdot B,\ B \to \cdot B,\ B \to \mathtt{x},$
  $S_2 \to \mathtt{x}C,\ C \to \mathtt{x}C, C \to \mathtt{x}\}$.

Useful Unary Operations

If W = $\begin{matrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{matrix}$ then

T(W) = $\begin{matrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{matrix}$ and H(W) = $\begin{matrix} a_{m,n} & \cdots & a_{m,2} & a_{m,1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2,n} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,n} & \cdots & a_{1,2} & a_{1,1} \end{matrix}$

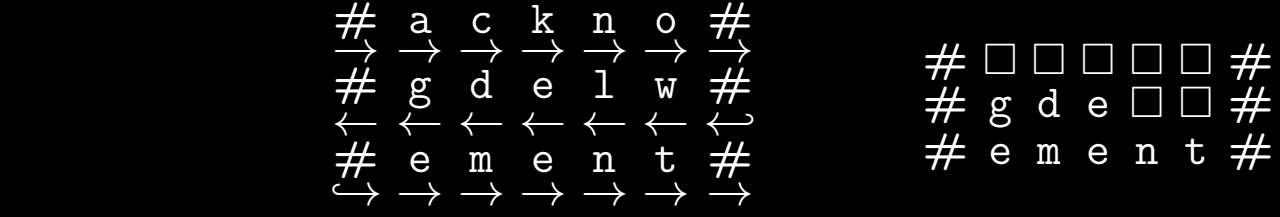
These operations are lifted to array languages and even to families of array languages in a natural way.

$T(\mathcal{L}_{\Sigma}(\mathrm{RMG}))$ = Any Guess ?

Boustrophedon Finite Automata

- Introduced by [H. Fernau et al 2015].

- Finite automata working on rectangular-shaped arrays (pictures). This automaton model reads the input, line after line, alters the direction, when the boundary is encountered.



Theorem 1. $T(\mathcal{L}_{\Sigma}(\mathsf{RMG})) = \mathcal{L}_{\Sigma}(\mathsf{BFA})$.

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(Regular : Regular) Array Grammars

A *(Regular : Regular) Array Grammar* (R:R)AG is defined by a 8-tuple $G = (S, V_N, V_I, \Sigma, P_N, P_I, \pi, \tau)$, where:

- V_N is a nonterminal alphabet with a distinctive start symbol $S \in V_N$;
  V_I is an intermediate alphabet, disjoint from V_N;
  Σ is a terminal alphabet, disjoint from $V_N \cup V_I$;

- P_N is a set of non-terminal rules
  either of the form $A \to XB$ (right-linear),
  or of the form $A \to BX$ (left-linear), where $A, B \in V_N$ and $X \in V_I$;

- P_I is a set of rules of the form $A \to X$, with $A \in V_N$ and $X \in V_I$;

- $\pi : V_I \to \mathcal{L}_\Sigma(\mathsf{RMG}) \cup \mathcal{L}_\Sigma(\mathsf{BFA})$;

- $\tau : P_N \to \{\ominus, \oplus\}$ such that $\tau(p_1) = \tau(p_2)$
  implies that p_1 is right-linear if and only if p_2 is right-linear.

(Regular : Regular) Array Grammars

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- $\pi : V_I \to \mathcal{L}_\Sigma(\mathsf{RMG}) \cup \mathcal{L}_\Sigma(\mathsf{BFA})$;

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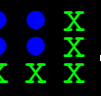
  implies that p_1 is right-linear if and only if p_2 is right-linear.

We denote the corresponding class of picture languages, as $\mathcal{L}((\mathsf{R} : \mathsf{R})\mathsf{AG})$ which was called as (Regular : Regular) Array Languages (R:R)AL [Siromoney et al 73].

Example

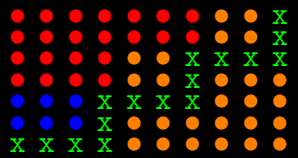
A (R:R)AG generating the staircase of x's of a fixed proportion is defined as $G = (S, V_N, V_I, \Sigma, P_N, P_I, \pi, \tau)$, where

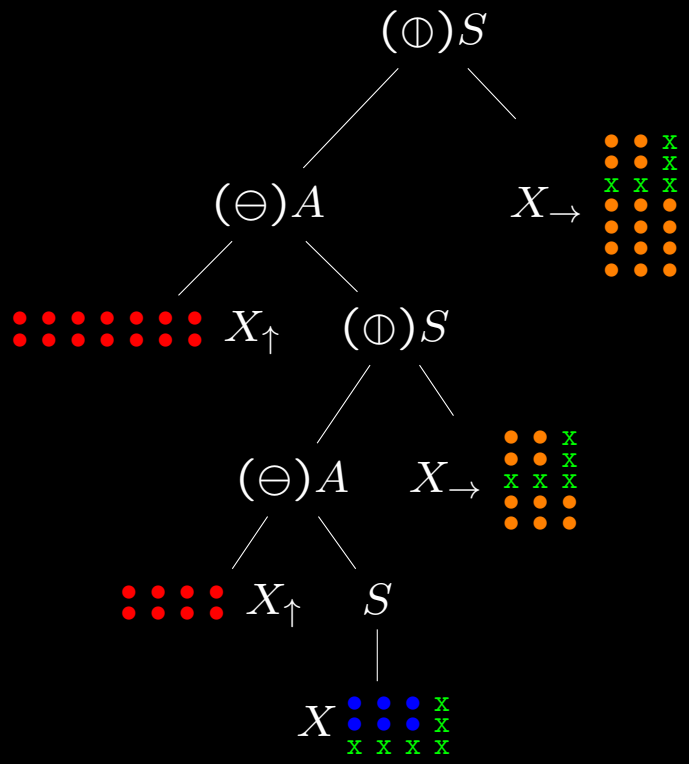
- $V_N = \{S, A\}$, $V_I = \{X_\uparrow, X_\rightarrow, X\}$, $\Sigma = \{\mathrm{x}, \bullet\}$,
  $P_N = \{S \to AX_\rightarrow, A \to X_\uparrow S\}$, $P_I = \{S \to X\}$,

- $\pi : V_I \to \mathcal{L}_\Sigma(\mathrm{RMG}) \cup \mathcal{L}_\Sigma(\mathrm{BFA})$ is given by
  $\pi(X_\uparrow) = \{(\bullet\bullet\bullet)^n \oplus (\bullet) \mid n \geq 1\}$,
  $\pi(X_\rightarrow) = \{(\bullet\bullet\mathrm{x}\,\bullet\bullet\mathrm{x}\,\mathrm{x}\,\mathrm{x}\,\mathrm{x}) \ominus (\bullet\bullet\bullet)_n \mid n \geq 1\}$ and $\pi(X) =$  .

- $\tau : P_N \to \{\ominus, \oplus\}$ is given by
  $\tau(S \to AX_\rightarrow) = \oplus$ and $\tau(A \to X_\uparrow S) = \ominus$.
  Here $\tau(S \to AX_\rightarrow) \neq \tau(A \to X_\uparrow S)$.

Staircase Example

To obtain  the corresponding tree:



Two Types of (R:R)AL

Definition 2. *An (R:R)AG G with $|\tau(P_N)| = 2$ is called* $\ominus$-left, $\oplus$-right

- *if $A \to BX \in P_N$ then $\tau(A \to BX) = \ominus$,*
- *if $A \to XB \in P_N$ then $\tau(A \to XB) = \oplus$.*

$\ominus$-left, $\oplus$-right (R:R)AG describes the language family $\mathcal{L}_{\ominus-\ell,\oplus-r}((\mathrm{R} : \mathrm{R})\mathrm{AG})$.

Definition 3. *An (R:R)AG G with $|\tau(P_N)| = 2$ is called* $\oplus$-left, $\ominus$-right

- *if $A \to BX \in P_N$ then $\tau(A \to BX) = \oplus$,*
- *if $A \to XB \in P_N$ then $\tau(A \to XB) = \ominus$.*

$\oplus$-left, $\ominus$-right (R:R)AG describes the language family $\mathcal{L}_{\oplus-\ell,\ominus-r}((\mathrm{R} : \mathrm{R})\mathrm{AG})$.

Note: Staircase Example is generated by some $\oplus$-left, $\ominus$-right (R:R)AG.

Results

Lemma 4.
$\mathcal{L}((\mathsf{R} : \mathsf{R})\mathsf{AG}) = \mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG}) \cup \mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG})$.

Lemma 5. *(a) A regular string language corresponds to a language of single-row arrays generated by RMG.*
(b) A regular string language corresponds to a language of single-column arrays generated by RMG.

Corollary 6. *(a) A regular string language corresponds to a language of single-row arrays accepted by BFA.*
(b) A regular string language corresponds to a language of single-column arrays accepted by BFA.

Remark 7. *A language of single-row (column) arrays is in both families* $\mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG})$ *and* $\mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG})$.

Results

News: $\mathcal{L}(\mathsf{RMG})$ is not closed under $\ominus$, $\mathcal{L}(\mathsf{BFA})$ is not closed under $\oplus$.

Consider the classes $\mathcal{L}(\mathsf{RMG}) \ominus \mathcal{L}(\mathsf{RMG})$ and $\mathcal{L}(\mathsf{BFA}) \oplus \mathcal{L}(\mathsf{BFA})$ and the rules $S \to X_1 A$ and $A \to X_2$ with
either $\pi(X_1), \pi(X_2) \in \mathcal{L}(\mathsf{RMG})$ and $\tau(S \to X_1 A) = \ominus$
or $\pi(X_1), \pi(X_2) \in \mathcal{L}(\mathsf{BFA})$ and $\tau(S \to X_1 A) = \oplus$.

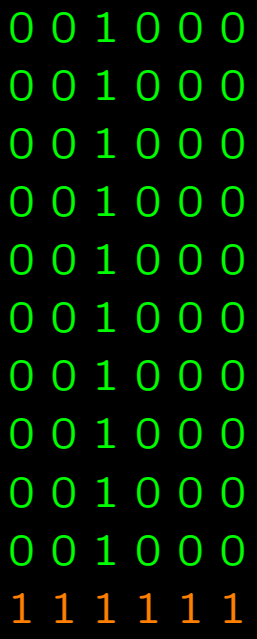
$(\mathcal{L}(\mathsf{RMG}) \ominus \mathcal{L}(\mathsf{RMG})) \subsetneq \mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG})$
and $(\mathcal{L}(\mathsf{BFA}) \oplus \mathcal{L}(\mathsf{BFA})) \subsetneq \mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG})$.

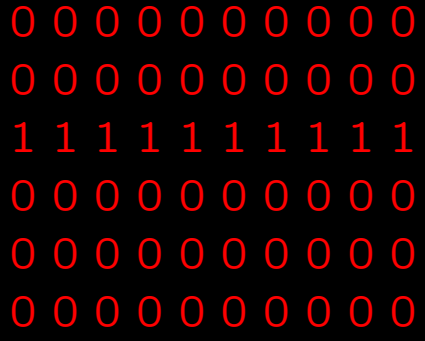
Lemma 8.

$$(\mathcal{L}(\mathsf{RMG}) \ominus \mathcal{L}(\mathsf{RMG})) \cup (\mathcal{L}(\mathsf{BFA}) \oplus \mathcal{L}(\mathsf{BFA}))$$

$$\subsetneq \mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG}) \cap \mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R} : \mathsf{R})\mathsf{AG}).$$

Results





Lemma 9. $(\mathcal{L}(\mathrm{RMG}) \ominus \mathcal{L}(\mathrm{RMG})) \subsetneq (\mathcal{L}(\mathrm{RMG}) \ominus \mathcal{L}(\mathrm{RMG}) \ominus \mathcal{L}(\mathrm{RMG}))$.

Corollary 10. $(\mathcal{L}(\mathrm{BFA}) \oplus \mathcal{L}(\mathrm{BFA})) \subsetneq (\mathcal{L}(\mathrm{BFA}) \oplus \mathcal{L}(\mathrm{BFA}) \oplus \mathcal{L}(\mathrm{BFA}))$.

Results

Recall: For $k, k' \in \mathbb{N}$, by W^k we denote the k-fold column-concatenation of the given array W, by W_k we denote the k-fold row-concatenation of W.

Theorem 11. *For each $k \geq 1$, we have:*
$\mathcal{L}(\mathsf{RMG})_k \subsetneq \mathcal{L}(\mathsf{RMG})_{k+1}$*, as well as* $\mathcal{L}(\mathsf{BFA})^k \subsetneq \mathcal{L}(\mathsf{BFA})^{k+1}$.

As (fixed) finite catenations of a single type can be expressed both by $\mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG})$ and by $\mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG})$, we conclude:

Corollary 12. *For any $k \geq 1$,*
$\mathcal{L}(\mathsf{RMG})_k \cup \mathcal{L}(\mathsf{BFA})^k \subsetneq \mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG}) \cap \mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG})$.

Results

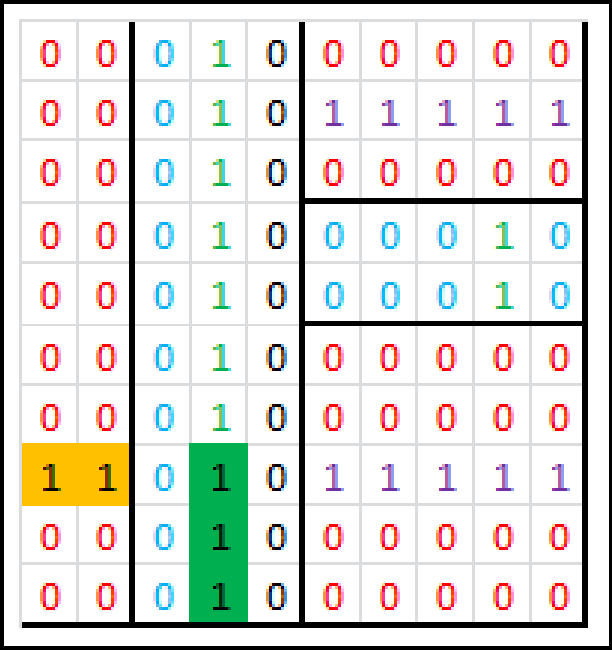
Theorem 13. $\mathcal{L}((\mathrm{R} : \mathrm{R})\mathrm{AG})$ *is not closed under union.*

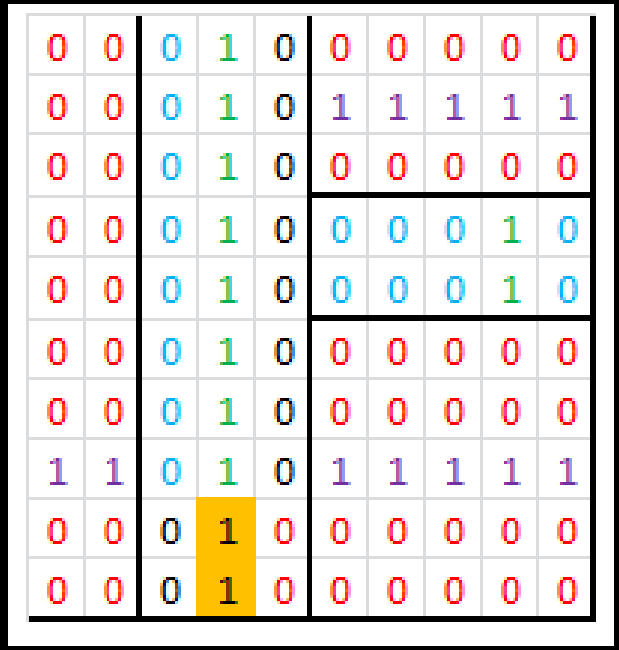
Proof Idea:

Recall $L = \{0\}^{+}_{+} \oplus \{1\}_{+} \oplus \{0\}^{+}_{+}$

0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0
0 0 1 0 0 0

Let $\widehat{L} = (T(L) \ominus L \ominus T(L)) \oplus L \oplus T(L)$. $\widehat{L}$ can be described by some $\oplus$-left, $\ominus$-right (R:R)AG (see end of page 125). Note that $H(\widehat{L}) = T(L) \oplus L \oplus (T(L) \ominus L \ominus T(L))$.





Lemma 14. $\mathcal{L}_{\ominus-\ell,\oplus-r}((\mathrm{R}:\mathrm{R})\mathrm{AG})$ *and* $\mathcal{L}_{\oplus-\ell,\ominus-r}((\mathrm{R}:\mathrm{R})\mathrm{AG})$ *are closed under union.*

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Regular Array Grammars

Isometric Approach: Σ^{++}
Non-Rectangular Arrays

Non-Isometric Approach: $\Sigma^{\substack{+\\+}}$
Rectangular Arrays

Isometric Regular Array Grammars

An *isometric regular array grammar (IRAG)* is defined by a 5-tuple $G = (N, \Sigma, P, S, \#)$, where:

- N - *nonterminal alphabet*; Σ - *terminal alphabet*;
  P - set of *rules*; $S \in N$ - *start symbol*; $\#$ - blank symbol;

- Every rule from P is of the form

$$\#A \to Ba \quad , \quad A\# \to aB \quad , \quad \begin{matrix} \# \\ A \end{matrix} \to \begin{matrix} B \\ a \end{matrix} \quad , \quad \begin{matrix} A \\ \# \end{matrix} \to \begin{matrix} a \\ B \end{matrix} \quad , \text{ or } \quad A \to a \quad ,$$

  where $A, B \in N$ and $a \in \Sigma$.

Example of an IRAG

IRAG using all four types of movements: Consider the array language L the set of all square pictures of diagonal lines from the upper left corner to the lower right corner where the elements in the diagonal are 1 and the other elements are 0.

The IRAG $G = (N, \Sigma, P, S, \#)$ such that $L_{Rect}(G) = L$ is defined as:

$N = \{S, A, B, C, H, E, F\}$, $\Sigma = \{0, 1\}$, $P = P_1 \cup P_2 \cup P_3$ where

$P_1 = \{1 : \begin{matrix}\# \\ S\end{matrix} \to \begin{matrix}A \\ 1\end{matrix}$, $2 : \begin{matrix}\# \\ A\end{matrix} \to \begin{matrix}A \\ 0\end{matrix}$, $3 : \#A \to B\,0$, $4 : \#B \to B\,0$, $5 : \begin{matrix}B \\ \#\end{matrix} \to \begin{matrix}1 \\ F\end{matrix}$,
$6 : \begin{matrix}C \\ \#\end{matrix} \to \begin{matrix}0 \\ C\end{matrix}$, $7 : C\# \to 0\,H$, $8 : H\# \to 0\,H$, $9 : \begin{matrix}\# \\ H\end{matrix} \to \begin{matrix}E \\ 0\end{matrix}$,
$a : \begin{matrix}\# \\ E\end{matrix} \to \begin{matrix}A \\ 1\end{matrix}$, $b : \begin{matrix}F \\ \#\end{matrix} \to \begin{matrix}0 \\ C\end{matrix}$, $c : S \to 1$ $\}$, $P_2 = \{d : F \to 0\}$ and $P_3 = \{f : E \to 1\}$.

$n = 5$:

$$\begin{array}{ccccccccc}
1 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_3 & 0 \\
\downarrow_5 & & & & & & & & \uparrow_2 \\
0 & & 1 & \leftarrow_4 & 0 & \leftarrow_3 & 0 & & 0 \\
\downarrow_b & & \downarrow_5 & & & & \uparrow_2 & & \uparrow_2 \\
0 & & 0 & & 1_f & & 0 & & 0 \\
\downarrow_6 & & \downarrow_b & & \uparrow_9 & & \uparrow_a & & \uparrow_2 \\
0 & & 0 & \rightarrow_7 & 0 & & 1 & & 0 \\
\downarrow_6 & & & & & & \uparrow_9 & & \uparrow_1 \\
0 & \rightarrow_7 & 0 & \rightarrow_8 & 0 & \rightarrow_8 & 0 & & 1
\end{array}$$

$n = 6$:

$$\begin{array}{ccccccccccc}
1 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_3 & 0 \\
\downarrow_5 & & & & & & & & & & \uparrow_2 \\
0 & & 1 & \leftarrow_4 & 0 & \leftarrow_4 & 0 & \leftarrow_3 & 0 & & 0 \\
\downarrow_b & & \downarrow_5 & & & & & & \uparrow_2 & & \uparrow_2 \\
0 & & 0 & & 1 & \leftarrow_3 & 0 & & 0 & & 0 \\
\downarrow_6 & & \downarrow_b & & \downarrow_5 & & \uparrow_a & & \uparrow_2 & & \uparrow_2 \\
0 & & 0 & & 0_d & & 1 & & 0 & & 0 \\
\downarrow_6 & & \downarrow_6 & & & & \uparrow_9 & & \uparrow_a & & \uparrow_2 \\
0 & & 0 & \rightarrow_7 & 0 & \rightarrow_8 & 0 & & 1 & & 0 \\
\downarrow_6 & & & & & & & & \uparrow_9 & & \uparrow_1 \\
0 & \rightarrow_7 & 0 & \rightarrow_8 & 0 & \rightarrow_8 & 0 & \rightarrow_8 & 0 & & 1
\end{array}$$

More Results

Lemma 15. *A language of single-row (column) arrays generated by some IRAG corresponds to a regular string language.*

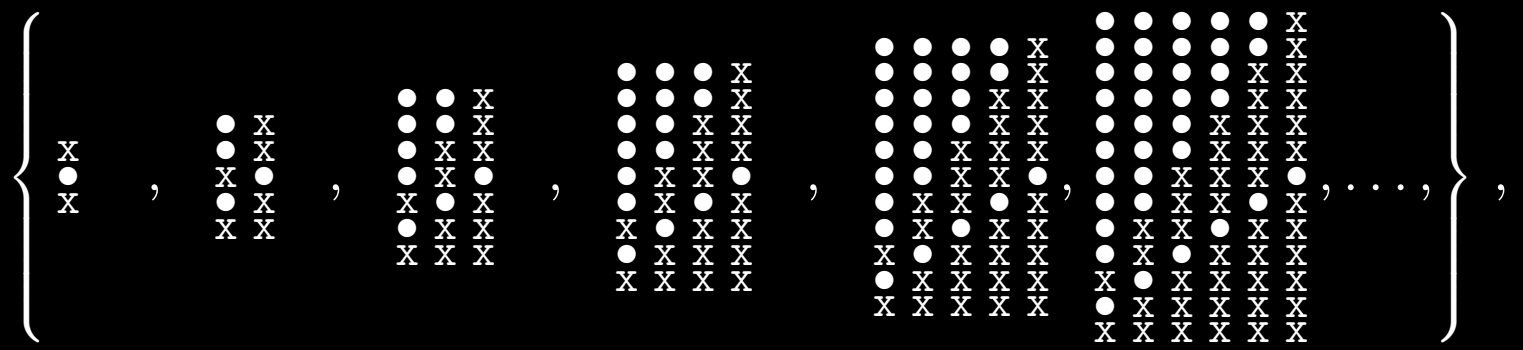
Definition 16. Semi-holes *are some blank symbol surrounded by three terminal symbols and one blank symbol on its four sides, or by two terminal symbols and two blank symbols.* Holes *are some blank symbols completely surrounded by terminal symbols.*

Lemma 17. *Each BFA can be simulated by some self-delimiting IRAG.*

Lemma 18. *Each BFA can be simulated by some IRAG that scans a picture with a fixed left border, starting in any corner and ending in a different corner of the array, assuming that the picture is big enough.*

Example for IRAG

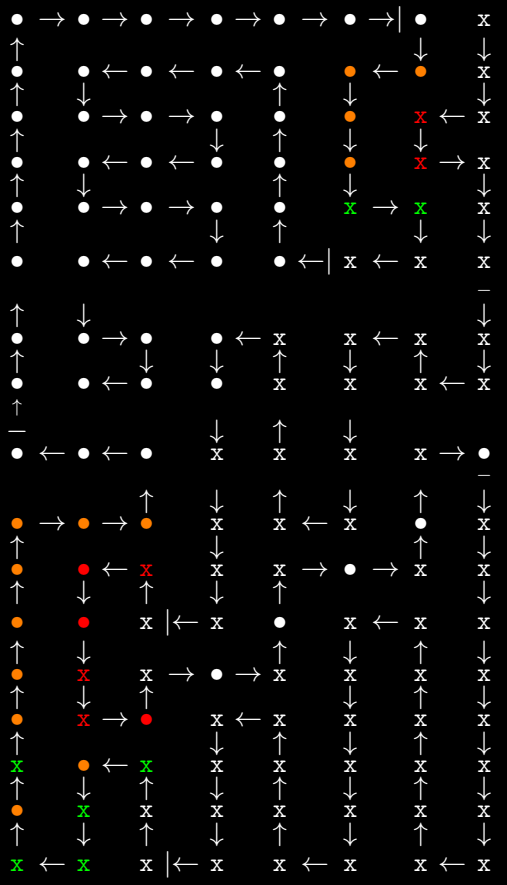
Consider the array language

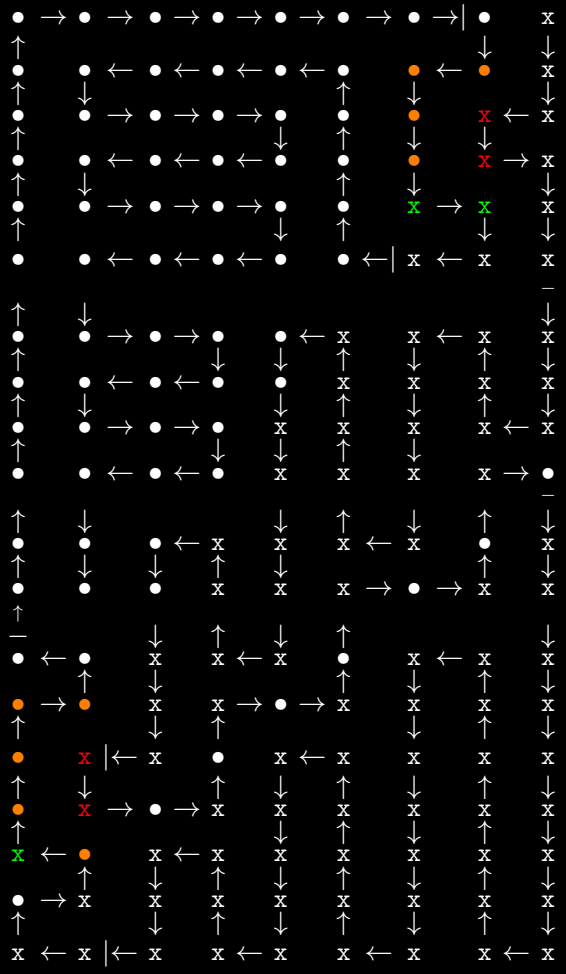
$L =$ 

as argued by [Siromoney et al 73] there is no (R:R)AG that can generate this language L.

Next Slide: Attempt to define an IRAG for L.

How to scan like IRAG: the picture M_8 (on the left) and M_9 (on the right)





More Results

Theorem 19. $\mathcal{L}((\mathsf{R}:\mathsf{R})\mathsf{AG}) \subsetneq \mathcal{L}_{Rect}(\mathsf{IRAG})$.

Theorem 20. *For a language of single-row (column) arrays L, the following are equivalent.*

- L *corresponds to a regular string language.*
- $L \in \mathcal{L}(\mathsf{RMG}) \cap \mathcal{L}(\mathsf{BFA})$.
- $L \in \mathcal{L}_{\ominus-\ell,\oplus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG}) \cap \mathcal{L}_{\oplus-\ell,\ominus-r}((\mathsf{R}:\mathsf{R})\mathsf{AG})$.
- L *is generated by some (R:R)AG.*
- L *is generated by some IRAG.*

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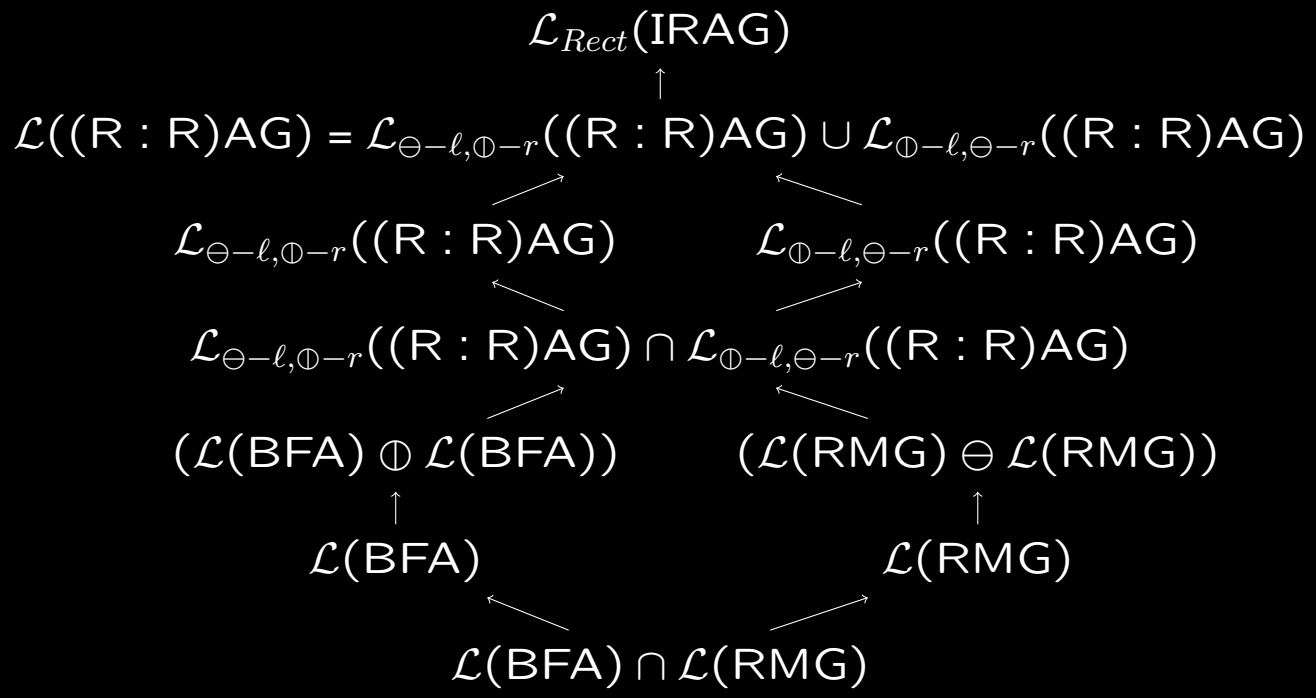

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